European Journal of Molecular & Clinical Medicine

ISSN 2515-8260 Volume 07, Issue 09, 2020

FIXED POINT THEOREMS FOR INTEGRAL TYPE F-CONTRACTIONS IN COMPLEX VALUED G_B-METRIC SPACES

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Abstract: In this paper, we obtain a unique fixed point theorem of integral type F-contractions in complex valued G_b-metrics paces.
Keywords: G-metric space, F-contraction, weaklycompatible.

1 Introduction and Preliminaries

In many branches of science, economics, computer science, engineering and the development of non-linear dynamics, the fixed point theory is one ofthe most important tool. In 1989, I. A. Bakhtin [1] introduced the contraction mapping principle in quasimetric spaces. In 2006, Mustafa and Sims introduced generalised metric spaces and extended fixed point theorems for contractive mappings in complete G-metric spaces. Abbas, Nazir and Vetro introduced common fixed point results for three maps in G-metricspaces.Vildan Uzturk introduced integral type F-contractions in partial metric spaces.In this paper, we obtain a unique fixed point theorem of integral type F-contractions in G-metric space which is generalised results of[6].

Definition 1.1 [4]Let X be a non-empty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{C}$ satisfies:
(CG_b1) $G(x, y, z) = 0$ if $x = y = z$;
(CG_b2) $0 \prec G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(CG_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(CG_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z;

(CG_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all x, y, z, a $\in$X.
Then,GiscalledacomplexvaluedG_b-metricand(X,G)iscalledacomplex valued G_b-metricspace.

Definition1.2 [4]Let(X,G)beacomplexvaluedG_b-metricspace.Then for any x, y, z $\in$X,

(i) $G(x, y, z) \leq s(G(x, x, y) + G(x, x, z))$,

(ii) $G(x, y, y) \leq 2sG(y, x, y)$.

Definition 1.3 [4] Let (X, G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$iscomplexvaluedG_b-convergenttoxifforeverya$\in\mathbb{C}$with $0 \prec a$, there exists k $\in\mathbb{N}$ such that $G(x, x_n, x_m) \prec a$ for all n, m $\geq$ k.

(ii) Asequence$\{x_n\}$iscalledcomplexvaluedG_b-Cauchyifforeverya$\in\mathbb{C}$with $0 \prec a$, there exists k $\in\mathbb{N}$such that $G(x_n, x_m, x_l) \prec a$ for all n,m,l$\geq$k.

(iii) IfforeverycomplexvaluedG_b-CauchysequenceiscomplexvaluedG_b-convergent in (X, G), then (X,G) is said to be complex valued G_b-complete.

Proposition 1.1 [4] Let (X, G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G_b-convergent to x if and only if $|G(x, x_n, x_m)| \to 0$ as n, m $\to \infty$.

Theorem1.1[4]Let(X,G)beacomplexvaluedG_b-metricspace,thenfor asequence$\{x_n\}$inXandpointx$\in$X,thefollowingareequivalent:

(i) $\{x_n\}$ is complex valued G_b-convergent tox.

(ii) $|G(x_n,x_n,x)| \to 0$asn$\to\infty$.

(iii) $|G(x_n,x,x)| \to 0$asn$\to\infty$.

(iv) $|G(x_m,x_n,x)| \to 0$asm,n$\to\infty$.

Theorem1.2[4]Let(X,G)beacomplexvaluedG_b-metricspaceand$\{x_n\}$ beasequenceinX.Then$\{x_n\}$iscomplexvaluedG_b-Cauchysequenceifand onlyif$|G(x_n,x_m,l)| \to 0$asn,m,l$\to\infty$.

Definition 1.4 Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying

(F1) F is strictly increasing. that is , $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$,
for all $\alpha, \beta \in \mathbb{R}_+$

(F2) For every sequence α_n in R_+ we have $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$

(F3) There exists a number k $\in$ (0,1) such that $\lim_{\alpha\to 0+} \alpha_k F(\alpha)=0$

Definition1.5 Let(X,G)beacomplexvaluedG_b-metricspace.Amapping T:X−−>XissaidtobeanF-contractionifthereexistsanumber$\tau>0$ such that $G(Tx, Ty, Tz) > 0 \to \tau + F(G(Tx,Ty,Tz)) \leq F(G(x,y,z))$ for all x,y,z $\in$X

Definition1.6[5]LetfandgbemapsformacomplexvaluedG_b-metric space(X,G)intoitself.Themapsfandgarecalledcompatiblemapsif there exists a sequence $\{x_n\}$ suchthat

$$\lim_{n\to\infty} G\,(fgx_n, gfx_n, gfx_n)=0 \text{ or } \lim_{n\to\infty} G\,(gfx_n, fgx_n, fgx_n)=0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t∈X..

Example1.1[5]LetX=[−1,1]andacomplexvaluedG_b-metriconXbe given asfollows:

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |x - z|^2$$

where s = 2 [4]. Define two mappings f, g : X → X by f (x) = xand g(x)=$x/3$ .If we consider a sequence $\{x_n\}$ = 1/2n , we obtain the following results:

$$\lim_{n\to\infty} G\,(fgx_n, gfx_n, gfx_n)=\lim_{n\to\infty} G\left(\frac{1}{6n},\frac{1}{6n},\frac{1}{6n}\right)=0$$

and

$\lim_{n\to\infty}\ fx_n=\lim_{n\to\infty}\frac{1}{2n}=0$ and $\lim_{n\to\infty}\ fx_n=\lim_{n\to\infty}\frac{1}{6n}=0$.

Therefore f and g are compatible maps.

2 MAIN RESULTS

Theorem2.1 LetfandgbeacompatiblemapsofacomplexvaluedG_b-metricspace(X,G)satisfyingf(x)⊆g(x).SupposethereexistsF∈Fand τ>0suchthatforall x,y∈XsatisfyingG(f_x,f_y,f_z)>0

$$\tau + F\left(\int_0^{G(f_x,f_y,f_z)} \phi(t)dt\right) \le F\int_0^{M(x,y,z)} \phi(t)dt \rightarrow (1)$$

whereM(x,y,z)=max{G(f_x,g_y,g_z),G(g_x,f_y,g_z),G(g_x,g_y,f_z)}

andφ: [0,∞)→[0,∞) isaLebesgueintegrablemappingwhichissummable,non- negative and for each μ >0,

$\int_0^{\mu} \phi(t)dt > 0$ .If

(i) F iscontinuous

(ii) f (x) and g(x) areclosed,

then f and g have a unique common fixed point inX.

Proof 2.1 Let x_0∈ X be a point in X.We can choose a point x_1 in X such that fx_0= gx_1. More generally, a point x_{n+1} can be chosen such that

$$y_n= fx_n= gx_{n+1},\ n = 0, 1, 2, \ldots$$

Step I: Prove that $G(y_n, y_{n+1}, y_{n+1}) \to 0$ as $n \to \infty$.

$$\tau + F\left(\int_0^{G(y_n,y_{n+1},y_{n+1})} \phi(t)dt\right) \leq \tau + F\int_0^{G(fx_n,fx_{n+1},fx_{n+1})} \phi(t)dt \rightarrow (2)$$
$$\leq F\int_0^{M(x_n,x_{n+1},x_{n+1})} \phi(t)dt$$

$M(x_n,x_{n+1},x_{n+1})=$
$\max\{G(fx_n,gx_{n+1},gx_{n+1}), G(gx_n,fx_{n+1},gx_{n+1}), G(gx_n,gx_{n+1},fx_{n+1})\}$
$=\max\{G(fx_n,fx_n,fx_n), G(fx_{n-1},fx_{n+1},fx_n), G(fx_{n-1},fx_n,fx_{n+1})\}$
$=\max\{0, G(fx_{n-1},fx_{n+1},fx_n), G(fx_{n-1},fx_n,fx_{n+1})\}$
$= G(fx_{n-1},fx_n,fx_{n+1})$

By the rectangular inequality of complex valued G_b-metric space, we get
$G(fx_{n-1},fx_n,fx_{n+1}) \leq s(G(fx_{n-1},fx_n,fx_{n+1}) + G(fx_n,fx_n,fx_{n+1}))$
$\leq s(G(fx_{n-1},fx_n,fx_n) + 2G(fx_n,fx_{n+1},fx_{n+1}))$

Hence by the above inequality, shows that

$$G(fx_n,fx_{n+1},fx_{n+1}) \leq \frac{s}{1-2s} G(fx_{n-1},fx_n,fx_n)$$

$$G(fx_n,fx_{n+1},fx_{n+1}) \leq q\ G(fx_{n-1},fx_n,fx_n)$$

Where $q=\frac{s}{1-2s}< 1$. If we consider the same procedure, we obtain

$$G(fx_n,fx_{n+1},fx_{n+1}) \leq q^n\ G(fx_0,fx_1,fx_1)$$

Therefore,foralln,m∈ℕ,n<m,wehavethefollowingsbytherectangular property
$G(y_n, y_m, y_m) \leq s[G(y_n, y_{n+1}, y_{n+1}) + s^2G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + s^mG(y_{m-1}, y_m, y_m)]$

$\leq(q^n+q^{n-1}+\cdots+q^{m-1})G(y_0,y_1,y_1)$

$\leq \frac{qn}{1-q}G(y_0,y_1,y_1)$

Taking limits as $n \to \infty$, we have $\lim_{n\to\infty} G(y_n, y_m, y_m) = 0$.
From (2)

$$F\left(\int_0^{G(fx_{n-1},fx_n,fx_{n+1})} \phi(t)dt\right) \leq F\int_0^{qG(fx_{n-2},fx_{n-1},fx_n)} \phi(t)dt - \tau \rightarrow (4)$$

Continuing this way,we have

$$F\left(\int_0^{G(fx_{n-2},fx_{n-1}fx_n)} \phi(t)dt\right) \leq F(\int_0^{q^2G(fx_{n-3},fx_{n-2},fx_{n-1})} \phi(t)dt) - \tau \rightarrow (5)$$

From (4) and (5)

$$F\left(\int_0^{G(fx_{n-1},fx_n,fx_{n+1})} \phi(t)dt\right) \leq F\int_0^{qG(fx_{n-2},fx_{n-1},fx_n)} \phi(t)dt - \tau$$

$$\leq F\left(\int_0^{q^2 G(fx_{n-3},fx_{n-2},fx_{n-1})} \phi(t)dt\right) - 2\tau$$

$$\cdots\cdots\cdots$$

$$\leq F\left(\int_0^{q^n G(y_0,y_1,y_1)} \phi(t)dt\right) - (n-1)\tau \rightarrow (6)$$

Then, it follows that

$F\left(\int_0^{G(y_n,y_{n+1},y_{n+1})} \phi(t)dt\right) = -\infty$

By F ∈ F,we have

$$\lim_{n\to\infty} G(y_n,y_{n+1},y_{n+1}) = 0$$

StepII: Now,weprovethat{ y_n }isG_b-Cauchysequence.

ByF∈F,there exists k ∈ (0, 1) suchthat

$\lim_{n\to\infty} (G(y_n,y_{n+1},y_{n+1}))^k F(G(y_n,y_n,y_{n+1}))=0$

By (6)

$(G(y_n,y_{n+1},y_{n+1}))^k\left(F\left(\int_0^{G(y_n,y_{n+1},y_{n+1})} \phi(t)dt\right)\right)-\left(F\left(\int_0^{G(y_0,y_1,y_1)} \phi(t)dt\right)\right)$

$$\leq -(n-1)(G(y_n,y_{n+1},y_{n+1}))^k \leq 0 \rightarrow (7)$$

Using the above inequality and(7)

$\lim_{n\to\infty} n(G(y_n, y_{n+1}, y_{n+1}))^k = 0$.Therefore, there exists n_1 ∈ℕ such that

$n(G(y_n, y_{n+1}, y_{n+1}))^k < 1$, for all n >$n_1$ or

$G(y_n, y_{n+1}, y_{n+1}) < \frac{1}{n^k}$

Let m,n ∈ℕ *with* m> n>n_1; Using triangular inequality , we have

$G(y_n,y_m,y_l) \leq G(y_n,y_{n+1},y_{n+1})+G(y_{n+1},y_m,y_l)$

$$\leq \sum_{i=n}^{\infty} G(y_n,y_{n+1},y_{n+1})$$

$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$$

As k ∈(0,1), the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges, so

$\lim_{n,m,l\to\infty} G(y_n,y_m,y_l)=0$

Thus,y_n isaCauchysequencein(X,G).Thereforey_n isaCauchysequencein (X,G_b).Since(X,G)iscompleteG-metric space,then(X,G_b) iscompletemetricspace.Then,thereexistsau∈Xsuchthat

$\lim_{n,m,l\to\infty} G_b(y_n,y_m,y_l)=0.$

Moreover,

$G(u,u,u)= \lim_{n,m\to\infty} G(y_n,y_m,u)= \lim_{n,m\to\infty} G(y_n,y_m,y_l)=0$.

Since

$y_n \rightarrow y$, then fx_n and $gx_n \rightarrow u$.

Step III: Wewillprovethatfandghaveacoincidepoint.Since(X,G) isacomplex valuedcompleteG$_b$-metricspace,thereisapointu∈Xsuch that

$\lim_{n\rightarrow\infty} y_n = \lim_{n\rightarrow\infty} fx_n = \lim_{n\rightarrow\infty} gx_{n+1} = u$

Since f or g is continuous, we can assume that g is continuous. so,

$\lim_{n\rightarrow\infty} gfx_n = \lim_{n\rightarrow\infty} ggx_{n+1} = gu$

Moreover, by the fact that f and g are compatible maps

$lim_{n\rightarrow\infty}\, G\,(fgx_n,\, gfx_n,\, gfx_n) = 0$ implies that

$\lim_{n\rightarrow\infty} fgx_n = gu$

$$\tau + F\left(\int_0^{G(f_x,f_y,f_z)} \phi(t)dt\right) \leq F\int_0^{M(x,y,z)} \phi(t)dt$$

whereM(x,y,z)=max{G(f_x,g_y,g_z),G(g_x,f_y,g_z),G(g_x,g_y,f_z)}

If we set x= gx_n;y= x_n;z= x_n

M(gx_n,x_n,x_n)=max{G(fgx_n, gx_n, gx_n),G(ggx_n, fx_n, gx_n),G(ggx_n, gx_n, fx_n)}

=max{G(gu,u,u), G(gu,u,u),G(gu,u,u)}

=G(gu,u,u)

$$\tau + F\left(\int_0^{G(fgxn,gxn,gxn)} \phi(t)dt\right) \leq F\int_0^{G(gu,u,u)} \phi(t)dt$$

$$\tau + F\left(\int_0^{G(gu,u,u)} \phi(t)dt\right) \leq F\int_0^{G(gu,u,u)} \phi(t)dt$$

This is a contradiction with $\tau > 0$.

Thus we have gu=u

Similarly, fu=u.

Step IV: Weshowuniquenessofcommonfixedpoint.Letwbeanother commonfixedpointoffandgandw≠u. Fromequation(2),wehave

$$\tau + F\left(\int_0^{G(u,w,w)} \phi(t)dt\right) \leq \tau + F\int_0^{G(fu,fw,fw)} \phi(t)dt$$

$$\leq F\int_0^{G(u,w,w)} \phi(t)dt$$

M(u, w, w) = max{G(fu, gw, gw), G(gu, fw,gw), G(gu, gw, fw)}

≤ max{G(u, w, w), G(u, w, w), G(u, w, w)}

= G(u, w, w)

$$\tau + F\left(\int_0^{G(u,w,w)} \phi(t)dt\right) \leq \tau + F\int_0^{G(u,w,w)} \phi(t)dt$$

which is a contradiction.

So, u = w.

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