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ON THE FAMILY OF HYPERBOLAS

$$w^2 - 6z^2 + 2aw - 12bz - 6b^2 = 0$$

K. Meena
Former VC, Bharathidasan University, Trichy, Tamil Nadu

S. Vidhyalakshmi
Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu

T. Mahalakshmi
Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu

M. A. Gopalan
Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamil Nadu

Abstract - The family of hyperbolas represented by the non-homogeneous binary quadratic equation $w^2 - 6z^2 + 2aw - 12bz - 6b^2 = 0 \; (a, b \neq 0)$ ***is considered to obtain its non-zero distinct integer solutions. A few fascinating relations among its solutions are exhibited. Construction of second order Ramanujan numbers and Pythagorean triples are illustrated.***

Keywords: Non-homogeneous quadratic, binary quadratic, positive pell equation, integer solutions, second order Ramanujan numbers, Pythagorean triples.

Mathematics Subject Classification: 11D09

INTRODUCTION

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-17]. In this communication, yet another interesting hyperbola given by $w^2 - 6z^2 + 2aw - 12bz - 6b^2 = 0$ isconsidered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola. Formulation of second order Ramanujan numbers and Pythagorean triples are illustrated.

METHOD OF ANALYSIS

The family of hyperbolas under consideration is

$$w^2 - 6z^2 + 2aw - 12bz - 6b^2 = 0 \tag{1}$$

Where a and b are both non-zero integers.

The completion of squares on the lefts of (1) leads to the positive pell equation

$$Y^2 = 6X^2 + a^2 \tag{2}$$

Where

$$Y = w + a \quad , \quad X = z + b \tag{3}$$

After performing some algebra, the general solution (X_{n+1}, Y_{n+1}) to (2) is given by

$$X_{n+1} = a f_n + \frac{5a}{2\sqrt{6}} g_n ,$$

$$Y_{n+1} = \frac{5a}{2} f_n + a\sqrt{6}\, g_n , \quad n = -1,0,1,......$$

Where

$$f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}$$

$$g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$$

In view of (3), the general solution (w_{n+1}, z_{n+1}) to (1) is obtained as

$$w_{n+1} = Y_{n+1} - a = \frac{5a}{2} f_n + a\sqrt{6} g_n - a$$

$$z_{n+1} = X_{n+1} - b = a f_n + \frac{5a}{2\sqrt{6}} g_n - b , \quad n = -1,0,1,.....$$

To know the nature of solutions, that is, whether the solutions to (1) are even or odd, one has to go for their numerical values. For simplicity and clear understanding, a few numerical values to z and w satisfying (1) are presented in Table 1 below:

Table 1: Numerical solutions

n	w_{n+1}	z_{n+1}
-1	$4a$	$2a-b$
0	$48a$	$20a-b$
1	$484a$	$198a-b$
2	$4800a$	$1960a-b$
3	$47524a$	$19402a-b$
4	$470448a$	$192060a-b$
5	$4656964a$	$1901198a-b$

Now, Table 2 below represents the nature of solutions:

Table 2: Nature of solutions

a	b	w_{n+1}	z_{n+1}
even	even	even	even
even	odd	even	odd
odd	even	even	even
odd	odd	even	odd

From the above table, one may observe that the values of w are always even where as the values of z are even or odd according as b is even or odd.

A few fascinating relations among the solutions are exhibited below:

(i) $\frac{2}{a} w_{2n}$ is a perfect square

$(ii)\, w_{n+3} - 10w_{n+2} + w_{n+1} = 8a\,, \quad n = -1,0,1,.....$

$(iii)\, z_{n+3} - 10z_{n+2} + z_{n+1} = 8b\,, \quad n = -1,0,1,......$

$(iv)\, 20w_{n+1} - 2w_{n+2} + 18a = 5z_{n+2} - 49z_{n+1} - 44b$

$(v)\, z_{n+3} - 97z_{n+1} - 96b = 40w_{n+1} - 4w_{n+2} + 36a$

- Expressions representing nasty numbers:
  - $6a(20w_{2n+2} - 2w_{2n+3} + 20a)$
  - $30a(99w_{2n+2} - w_{2n+4} + 108a)$
  - $6a(5z_{2n+3} - 49z_{2n+2} - 44b + 2a)$
  - $3a(z_{2n+4} - 97z_{2n+2} - 96b + 4a)$
- Expressions representing cubical integers:
  - $\frac{1}{2a}(z_{3n+5} - 97z_{3n+3} + 3z_{n+3} - 291z_{n+1} - 384b)$
  - $\frac{1}{a}(20w_{3n+3} - 2w_{3n+4} + 60w_{n+1} - 6w_{n+2}) + 72$
  - $\frac{1}{a}(5z_{3n+4} - 49z_{3n+3} + 15z_{n+2} - 147z_{n+1} - 176b)$
- Let $z = z_0$, $w = w_0$ be any known solution to (1).

  Then, each of the following values of z and w also satisfies (1).
  - $z = 49z_0 - 20w_0 - 20a + 48b$
    $w = 120z_0 - 49w_0 - 50a + 120b$
  - $z = 5z_0 - 2w_0 - 2a + 4b$
    $w = 12z_0 - 5w_0 - 6a + 12b$
  - $z = 2w_0 - 5z_0 + 2a - 6b$
    $w = 5w_0 - 12z_0 + 4a - 12b$
- From the values of w_{n+1} , z_{n+1} , one may generate second order Ramanujan numbers.

Illustration 1:

Let $a = 5$, $b = 2$

Now, $z_0 = 8 = 1*8 = 2*4$ (*)

We write from the above relation (*)

$$(1+8)^2 + (4-2)^2 = (8-1)^2 + (2+4)^2$$

$$\Rightarrow 9^2 + 2^2 = 7^2 + 6^2 = 85$$

$\therefore$ 85 is the second order Ramanujan number with base numbers, namely 9,2,7,6 as real integers.

Illustration 2:

One may also write from (*)

$$(8+i)^2 + (4-2i)^2 = (8-i)^2 + (4+2i)^2 = 75$$

Here 75 is the second order Ramanujan number with base numbers, namely, $8+i$, $4-2i$, $8-i$, $4+2i$ as Gaussian integers.

- When a is a perfect square, from the value of w_{2n} , $n = -1,0,1,....$
  one may generate Pythagorean triple.

Illustration 3:

Let $a = \alpha^2$

Then, $w_0 = 4\alpha^2 = 1 * 4\alpha^2 = \alpha * 4\alpha = 2\alpha * 2\alpha = 2\alpha^2 * 2$

(i) Consider the relation

$$1 * 4\alpha^2 = 2\alpha * 2\alpha$$

We write

$$\left(1+4\alpha^2\right)^2 + (2\alpha - 2\alpha)^2 = \left(1-4\alpha^2\right)^2 + (2\alpha + 2\alpha)^2$$

$$\left(1+4\alpha^2\right)^2 = \left(1-4\alpha^2\right)^2 + (4\alpha)^2 \qquad \text{(I)}$$

(ii) Consider the relation

$$\alpha * 4\alpha = 2\alpha * 2\alpha$$

We write

$$(\alpha + 4\alpha)^2 + (2\alpha - 2\alpha)^2 = (\alpha - 4\alpha)^2 + (2\alpha + 2\alpha)^2$$

$$\Rightarrow \quad (5\alpha)^2 = (-3\alpha)^2 + (4\alpha)^2 \qquad \text{(II)}$$

(iii) Consider the relation

$$2\alpha^2 * 2 = 2\alpha * 2\alpha$$

We write

$$\left(2\alpha^2 + 2\right)^2 + (2\alpha - 2\alpha)^2 = \left(2\alpha^2 - 2\right)^2 + (2\alpha + 2\alpha)^2$$

$$\Rightarrow \quad \left(2\alpha^2 + 2\right)^2 = \left(2\alpha^2 - 2\right)^2 + (4\alpha)^2 \qquad \text{(III)}$$

Note that (I), (II) and (III) represent Pythagorean triples.

CONCLUSION

As hyperbolas are rich in variety, one may consider other choices of hyperbolas for determining their integer solutions with suitable properties.

REFERENCES

1. David M. Burton, elementary Number Theory, Tata MC Graw hill Publishing company, limited New Delhi -2000.
2. Telang S.J, Number Theory, Tata Mc Graw Hill publishing, Company limited New Delhi-2000.
3. K. Meena, M.A. Gopalan, T. Swetha, “On the negative pellequation $y^2 = 40x^2 - 4$”, International Journal of Emerging Technologies in Engineering Research (IJETER)volume 5, Issue 1, January(2017),(6-11).
4. K.Meena, S. Vidhyalakshmi, G. Dhanalakshmi, “On the negative pell equation $y^2 = 5x^2 - 4$”, Asian Journal of Applied Science and Technology(AJAST)volume 1, Issue 7, pages 98-103, August 2017.
5. M.A.Gopalan,V.Geetha,S.Sumithra, “Observations on the Hyperbola $y^2 = 55x^2 - 6$”, International Research Journal of Engineering and Technology (IRJET) volume: 02 Issue: 04\ July-2015, (1962-1964).
6. M.A. Gopalan, S. Vidhyalakshmi, T.R. Usharani, M. Arulmozhi, “On the negative pell equation $y^2 = 35x^2 - 19$”, International Journal of Research and Review, vol.2; Issue:4; April 2015, pages 183-188.

7. M.A. Gopalan, S.Vidhyalakshmi, J. Shanthi, D. Kanaka, "On the Negative Pell Equation $y^2 = 15x^2 - 6$", Scholars Journal of Physics, Mathematics and Statistics, Sch.J.Phys.Math.Stat.2015; 2(2A):123-128.
8. K. Meena, S.Vidhyalakshmi,A.Rukmani, " On the Negative pell Equation $y^2 = 31x^2 - 6$", Universe of Emerging Technologies and Science,2015;2(12) , 1-4.
9. M.A.Gopalan,S.Vidhyalakshmi,V.Pandichelvi,P.Sivakamasundari,C.Priyadharsini., " On The Negative pell Equation $y^2 = 45x^2 - 11$", International Journal of Pure Mathematical Science,2016; volume-16;30-36.
10. S. Vidhyalakshmi, M.A. Gopalan, E. Premalatha,S. Sofiachristinal, " On The Negative pell Equation $y^2 = 72x^2 - 8$" , International Journal of Emerging Technologies in Engineering Research (IJETER),volume 4,Issue 2,Feburary 2016,25-28.
11. M. Devi, T.R. Usharani, "On the Binary Quadratic Diophantine Equation $y^2 = 80x^2 - 16$", Journal of Mathematics and Informatics ,vol.10,2017,75-81.
12. R. Suganya, D. Maheswari, " On the Negative pellian Equation $y^2 = 110x^2 - 29$", Journal of Mathematics and Informatics, vol.11,2017,63-71.
13. P. Abinaya, S. Mallika, " On the Negative pellian Equation $y^2 = 40x^2 - 15$", Journal of Mathematics and Informatics, vol.11,2017,95-102.
14. S. Vidhyalakshmi, M.A. Gopalan, T. Mahalakshmi, " On the Negative pell Equation $y^2 = 40x^2 - 15$", IJESRT, vol.7, Issue11,November 2018, 50-55.
15. S. Vidhyalakshmi, T. Mahalakshmi, "On the Negative pellEquation $y^2 = 15x^2 - 14$", IJRASET, vol.7, Issue III, March 2019, 891-897.
16. M.A. Gopalan, T. Mahalakshmi,K.Sevvanthi "On The Positive Pell Equation $y^2 = 35x^2 + 29$", International Research Journal of Engineering and Technology (IRJET), Volume 6, Issue 3, Pages 1829-1837, March 2019.
17. S. Vidhyalakshmi, M.A. Gopalan, T. Mahalakshmi, " A Study On The Hyperbola $3x^2 + 7xy + 3y^2 - 13x - 13y + 9 = 0$", Global Journal Of Engineering Science And Researches (GJESR), Volume 6, Issue 2, Pages 61-65, February 2019.