A Retrial Queuing System Operating in a Random Environment Subject to Catastrophes

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Abstract: ***An one server retrial lineup system operating in a uncertain environment subject to server failure along with repair is analysed. The environment is in any one of the m + 1 states 0, 1, 2,…, m. The environmental state 0 denotes to the state that the server is undergoing repair. The mean*** $\frac{1}{\eta_0}$ ***is exponential to the repair time. Throughout repair time, customer is not allowed to join the orbit. Instantaneously once the repair, the system drives to phase i, i ≥ 1 with probability*** q_i***, where*** $\sum_{i=1}^{m} q_i=1$***. When the environment is in stage i ≥1, the model acts like an*** $M(\lambda_i)/M(\mu_i)/1$ ***queue with service and arrival rate*** μ_i ***and*** λi ***respectively . There is no waiting room and any arriving customer who discovers the server idle joins for service; else (in case of busy server) goes to an orbit of infinite capacity and retries for service with rate v. The system resides in phase*** i ***for a random intermission of time is exponential with mean*** $\frac{1}{\eta_i}$ ***and at the end of the sojourn period, a catastrophe occurs washing out the customers (if any) in the orbit and also the customer (if any) undergoing service and the system moves to phase 0: The system steady-state behaviour is derived.***

Keywords: Queueing System, Retrial, Catastrophe, Repair.
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1. INTRODUCTION

We see that, the Queueing systems working on randomly occurring tragedies have been studied by, Sengupta [1], Yechiali [2], Chakravarthy [3], Krishna Kumar et al. [4], Sudhesh [5], Paz and Yechiali [6] and Udayabaskaran and Dora Pravina [7]) . The steady-state behaviour of an M/M/1 model queue operating in uncertain environment subject to disasters where the underlying environment is described by a n-phase continuous-time Markov chain have been analysed by Paz and Yechiali [6]. Udayabaskaran and Dora Pravina [7] have analysed the time-dependent behaviour model of Paz and Yechiali [6]. However, to our knowledge, retrial queueing systems operating in random environment and subject to randomly occurring disasters have not been studied so far in literature. The purpose of the present paper is to perform a stochastic analysis of a retrial queueing system operating in an uncertain conditions subject to catastrophes. We find the steady- state probability distribution of the queueing system.

The paper is organized as follows: the model of a retrial queueing system have been analysed in section 2. The time-dependent probabilities of the system equations will be discussed in section 3. Section 4 obtains obvious expressions in the steady-state probabilities.

2. EXPLANATION OF THE MODEL

We study a single server queuing system operating in an uncertain condition. The server fails due to the occurrence of catastrophes and it is immediately taken for repair. The mean $\frac{1}{\eta_0}$ is exponential to the repair time. As in Paz and Yechiali [3], we assume that the environment is in any one of the m+1 states 0, 1, 2,…, m. The environmental state 0 corresponds to the state that the server is undergoing repair. Instantaneously once repair, the server returns to work immediately without delay in state i, i ≥ 1 with probability q_i where $\sum_{i=1}^{m} q_i = 1.$ when it is in state i, customers arrive to externally to the system from outside with Poisson process with rate $\lambda_i, i = 0,1,2,\ldots,m+1$. These customers are called prime customers. There is no waiting room in the system and any prime customer who knows the server is busy immediately will move to an infinite capacity orbit and retries later for getting assistance. Customers in the orbit follow the classical retrial policy with rate nv, where n is the number of people in the orbit retrying for service. The system resides in phase i for a uncertain time interval which is exponentially $\frac{1}{\eta_i}$ as mean and at the instant of the end of the living period in phase i a disaster occurs washing out all the customers and the system goes to phase 0. During phase 0, no customer is allowed to join the orbit. When the conditions is in phase $i \geq 1$; the system performs like an $M(\lambda_i)/M(\mu_i)/1$ queue with arrival rate λ_i and service rate μ_i. Let S(t) be the state of the server (0 for undergoing repair, 1 for idle at service and 2 for actively serving) at time t, Let the state of the conditions at time t be E(t); and the number of customers in the orbit at time t be $X(t)$). Let $Z(t) = (S(t), E(t), X(t))$. Then the three-dimensional $\{Z(t) : t \geq 0\}$ is Markov. The state space of the process is known by $\Omega = \{(i,j,k) | i = 0,1,2;\ j = 0,1,2,\ldots.m;\ k = 0,1,2,\ldots.\}$. We assume that a catastrophe has just occurred at time t = 0: Then, we have $Z(0) = (0,0,0)$.We define the probability distribution of $(S(t), E(t), X(t))$ by

$$p(i,j,k,t) = Pr[Z(t) = (i,j,k) | Z(0) = (0,0,0)]\ . \tag{1}$$

In the next section, we derive the governing integral equations for p (i, j, k, t).

3. GOVERNING EQUATIONS

Standard probabilistic arguments yield the following integral equations for $p(i,j,k,t)$.

Case (i) $(0,0,0) \in \Omega$: For the system to be in state $(0,0,0)$ at time t, one of the following equally exclusive and exhaustive events should occur: (a) No event has taken place up to time t. (b) The system was in state (i,j,k) or $(2,j,k)$, $j = 0,1,2,\ldots.m$; $k = 0,1,2,\ldots.$ at time u ∈ (0, t),a catastrophe occurred in $(u, u+\Delta)$ and no event has occurred thereafter up to time t: Consequently, we get

$$p(0,0,0,t) = e^{-\eta_0 t} + \sum_{j=1}^{m}\sum_{k=0}^{\infty}\int_0^t [p(1,j,k,u) + p(2,j,k,u)]\eta_j e^{-\eta_0(t-u)}du. \tag{2}$$

$Case\ (ii)$ (1,j, 0) ∈ Ω , $j = 0,1,2,\ldots.m$:

$$p(1,j,0,t) = \int_0^t [p(1,j,k,u) + p(2,j,k,u)]\eta_j e^{-\eta_0(t-u)}du. \tag{3}$$

$Case\ (iii)$ $(1,j,k) \in \Omega$, $j = 0,1,2,\ldots.m$; $k = 0,1,2,\ldots.$:

$$p(1,j,k,t) = \int_0^t [p(2,j,k,u)\,\mu_j e^{-(\lambda_j+\eta_j+v)(t-u)}du. \tag{4}$$

$Case\ (iv)\ (2, j, 0)\ \in \Omega\ , j\ =\ 0, 1, 2, \ldots .m$:

$$p(2, j, 0, t) = \int_0^t [p(1, j, 0, u)\ \lambda_j + p(1, j, 1)\ e^{-(\lambda_j + \mu_j + \eta_j)(t-u)} du \quad (5)$$

$Case\ (v)\ (2, j, k)\ \in \Omega\ , j\ =\ 0, 1, 2, \ldots .m\ ;\ k\ =\ 0, 1, 2, \ldots$:

$$p(2, j, k, t) = \int_0^t [\{p(2, j, k-1, u) + p(1, j, k, u)\ \}\lambda_j + p(1, j, k+1, u)\mathrm{v}]\ e^{-(\lambda_j + \mu_j + \eta_j)(t-u)} du \quad (6)$$

Representing the Laplace transform of p(i, j, k, t) by p*(i, j, k, s), equations (2) - (7) yield

$$(s + \eta_0) p^*(0, 0, 0, s) = 1 + \sum_{j=1}^{m} \sum_{k=0}^{\infty} p^*(1, j, k, s) + p^*(2, j, k, s)]\eta_j; \quad (7)$$

$$\left(s + \lambda_j\ + \eta_j\ \right) p^*(1, j, 0, s) = p^*(0,0,0, s)\eta_0\, q_j\ + p^*(2, j, 0, s)\mu_j, j\ =\ 0, 1, 2, \ldots .m\ ; \quad (8)$$

$$\left(s + \lambda_j\ + \eta_j\ + v\right) p^*(1, j, k, s) = p^*(2, j, k, s)\mu_j, j\ =\ 0, 1, 2, \ldots .m\ ; k\ =\ 0, 1, 2, \ldots \quad (9)$$

$$\left(s + \lambda_j\ + \mu_j + \eta_j\ \right) p^*(2, j, 0, s) = p^*(1, j, 0, s)\}\lambda_j + p^*(1, j, 1, s)\}v\ , j\ =\ 0, 1, 2, \ldots .m; \quad (10)$$

$$\left(s + \lambda_j\ + \mu_j + \eta_j\ \right) p^*(2, j, k, s) = \{p^*(2, j, k-1, s) + p^*(1, j, k, s)\}\lambda_j +\ p^*(1, j, k+1, s)\}v,$$
$$j\ =\ 0, 1, 2, \ldots .m\ ; k\ =\ 0, 1, 2, \ldots \quad (11)$$

4. STEADY-STATE SOLUTION

Steady-state probabilities are defined by

$$\pi(i, j, k) = \lim_{t\to\infty} p(i, j, k, t), i = 1{,}2;\ j\ =\ 0, 1, 2, \ldots .m; \qquad k\ =\ 0, 1, 2, \ldots \quad (12)$$

By Final Value Theorem of Laplace transform,

$$\pi(i, j, k) = \lim_{t\to\infty} sp^*(i, j, k, s), i = 1{,}2;\ j\ =\ 0, 1, 2, \ldots .m\ ; k\ =\ 0, 1, 2, \ldots$$

Consequently, equations (7) - (11) yield the following equations:

$$\eta_0\, \pi(0{,}0{,}0) = \sum_{j=1}^{m} \sum_{k=0}^{\infty} [\pi(1, j, k) + \pi(2, j, k)]\eta_j; \quad (13)$$

$$(\lambda_j + \eta_j)\pi(1, j, 0) = \pi(0{,}0{,}0)\eta_0\, q_j\ + \pi(2, j, 0)\mu_j, j\ =\ 0, 1, 2, \ldots .m; \quad (14)$$

$$(\lambda_j + \eta_j + v)\pi(1, j, k) = \pi(2, j, k)\mu_j, j\ =\ 1, 2, \ldots .m; k = 1{,}2, \ldots ; \quad (15)$$

$$(\lambda_j + \mu_j + \eta_j)\pi(2, j, 0) = \pi(1, j, 0)\lambda_j + \pi(1, j, 1)v, j\ =\ 1, 2, \ldots .m; \quad (16)$$

$$(\lambda_j + \mu_j + \eta_j)\pi(2, j, k) = [\pi(2, j, k-1)\lambda_j + \pi(1, j, k)]\lambda_j + \pi(1, j, k+1)v, j\ =\ 1, 2, \ldots .m;$$
$$k = 1{,}2, \ldots ; \quad (17)$$

Equations (13)-(17) can be solved by the technique of partial generating functions. These functions are defined by

$$\prod_{lj}(u) = \sum_{k=0}^{\infty} \pi(l, j, k)\, u^k, l = 1{,}2;\ j\ =\ 1, 2, \ldots .m. \quad (18)$$

Equations (14) and (15) yield

$$(\lambda_j + \mu_j + \eta_j) \prod_{lj}(u) = \mu_j \prod_{2j}(u) +\ \pi(0{,}0{,}0)\eta_0 q_j\ + v\, \pi(1, j, 0), j\ =\ 1, 2, \ldots .m \quad (19)$$

Equations (16) and (17) yield

$$(\lambda_j\, (1-u) + \mu_j + \eta_j) \prod_{2j}(u) = (\lambda_j + \frac{v}{u}) \prod_{2j}(u) +\ uv\pi(1, j, 0), j\ =\ 1, 2, \ldots .m \quad (20)$$

Equation (19) yields

$$\prod_{2j}(u) = \frac{(\lambda_j + \eta_j + v) \prod_{1j}(u) + \pi(0{,}0{,}0)\eta_0 q_j + v\, \pi(1, j, 0)}{\mu_j}, j\ =\ 1, 2, \ldots .m. \quad (21)$$

Equations (20) and (21) yield

$$\prod_{1j}(u) = \frac{(\alpha_j\, u - \lambda_j\, u^2)\pi(0{,}0{,}0)\eta_0 q_j + \{(\alpha_j\, u - \lambda_j\, u^2) - \mu_j\}v\, \pi(1, j, 0)}{-\lambda_j \beta_j u^2 + (\beta_j \alpha_j - \lambda_j \mu_j)u - \mu_j v}, \quad (22)$$

Where $\alpha_j\ = \lambda_j + \mu_j + \eta_j$ and $\beta_j\ = \lambda_j + \eta_j + v$. Using Rouche's theorem, equation (22) yields

$$\pi(1, j, 0) = \frac{\xi_1(\alpha_j - \lambda_j \xi_1)\eta_0 q_j}{\mu_j - \xi_1(\alpha_j - \lambda_j \xi_1)} \pi(0{,}0{,}0), j\ =\ 1, 2, \ldots .m. \quad (23)$$

where $\xi_1 \in (0{,}1)$ is the positive root of the quadratic equation

$$\lambda_j \beta_j\, u^2 - \left(\beta_j \alpha_j - \lambda_j \mu_j\right)u + \mu_j v = 0. \quad (24)$$

The other root ξ_2 of (24) lies outside (0, 1). Expanding (22) as a power series and equating like powers of u, the probabilities $\pi(1,j,k), k = 2, 3, \ldots$ can be obtained in terms of $\pi(0,0,0)$ Using $\pi(1,j,k)$ in (14), we can obtain $\pi(2,j,k)$. Using the total probability law

$$\pi(0,0,0) + \sum_{j=1}^{m} \sum_{k=0}^{\infty} [\pi(1,j,k) + \pi(2,j,k)] = 1,$$

We can obtain $\pi(0,0,0)$ explicitly.

5. CONCLUSION

Here, we governed the probabilities of steady-state for a lone server retrial lineup system operating in a uncertain environment subject to server failure and repair.

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