Some New Results For The Partitions With Designated Summands

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Abstract. Lin introduced and studied two new partition functions $PD_t(n)$ and $PDO_t(n)$, which count the number of designated summands in two partition functions $PD(n)$ and $PDO(n)$, respectively. In this paper, we prove the generating functions for $PDO_t(48n)$, $PDO_t(48n+14)$, $PDO_t(48n+16)$, $PDO_t(48n+22)$ and $PDO_t(48n+46)$. Also, we find some new congruences and infinite families of congruences modulo 16 and modulo 72 for $PDO_t(n)$.
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1. Introduction

MacMohan [1] introduced and examined a new class of partition function in which some parts are tagged and he studied these partition functions with exactly k different sizes.
In [2], Andrews, Lewis and Lovejoy studied the partitions with designated summands. In these partitions, exactly one part is tagged or designated among the parts with same magnitude. For example, there are ten partitions of 4 with designated summands, namely,
$4'$, $3'+1'$, $2'+2$, $2+2'$, $2'+1'+1$, $2'+1+1'$, $1'+1+1+1$, $1+1'+1+1$, $1+1+1'+1$, $1+1+1+1'$.
$PD(n)$ denotes the total number of partitions of n with tagged parts and $PDO(n)$ denotes the total number of partitions of n with tagged parts, where all parts are odd. Therefore, in the above example, $PD(4)=10$ and $PDO(4)=5$.
Chen, Ji, Jin and Shen [3] proved the generating function of $PD(3n)$ and $PD(3n+2)$ which implies the congruence given in [2].
Using the generating function of $PD(3n)$ and $PD(3n+2)$ defined in [3], Xia [4] proved some infinite families congruences modulo 9 and modulo 27 for $PD(n)$.
In [5], Lin introduced new partition functions $PD_t(n)$ and $PDO_t(n)$, where $PD_t(n)$ and $PDO_t(n)$ count the total number of designated summands in $PD(n)$ and $PDO(n)$ respectively. He established the generating functions for $PD_t(n)$ and $PDO_t(n)$ as

$$\sum_{n=0}^{\infty} PD_t(n)q^n = \frac{1}{2}\left(\frac{f_3^5}{f_1^3 f_6^2} - \frac{f_6}{f_1 f_2 f_3}\right)$$

and

$$\sum_{n=0}^{\infty} PDO_t(n)q^n = \frac{q f_2 f_3^2 f_{12}^2}{f_1^2 f_6},$$

where $f_k = (q^k;q^k)_\infty$ and $(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1})$.
Adansie, Chern and Xia [6] generalized Lin's conjecture $PD_t(27n+6) \equiv PD_t(27n+21) \equiv 0 (mod\ 9)$.
Baruah and Kaur [7] proved conjectures proposed by Lin for the congruences modulo 8 in [5]. Also, they proved some new congruences for $PD_t(n)$.
Recently, Chern and Hirschhorn [8] gave elementary proofs for $PD(n)$, $PDO(n)$, $PD_t(n)$ and $PDO_t(n)$ introduced by Andrews et al. [2] and Chen et al. [3]. They simplified the formulas for $\sum_{n\geq 0} PD(3n)q^n$ and $\sum_{n\geq 0} PD(3n+1)q^n$ and improved some congruences of Lin [5] for modulo 9 and modulo 27.

Very recently, we [9,10] proved the generating function for $PDO_t(16n)$ and $PDO_t(16n+8)$. In Theorem 1.1, we further extend these results and prove the generating functions for $PDO_t(48n)$ and $PDO_t(48n+16)$. Also, we prove a congruence modulo 72 for $PDO_t(n)$.

Theorem 1.1. For $n \geq 0$, we have

$$\sum_{n=0}^{\infty} PDO_t(48n+16)q^n \equiv 36f_2\psi(q^3)(mod\ 72) \quad , \tag{1.1}$$

$$\sum_{n=0}^{\infty} PDO_t(48n)q^n \equiv 36f_2f(q,q^2)\ (mod\ 72) \quad , \tag{1.2}$$

$$PDO_t(48n+32) \equiv 0(mod\ 72) \quad . \tag{1.3}$$

In the Theorem 1.2, we prove the generating functions for $PDO_t(48n+22)$, $PDO_t(48n+46)$ and $PDO_t(48n+224)$. Also, we prove some congruences modulo 16 for $PDO_t(n)$.

Theorem 1.2. For $n \geq 0$, we have

$$PDO_t(48n+6) \equiv 0(mod\ 16) \quad , \tag{1.4}$$

$$PDO_t(48n+38) \equiv 0(mod\ 16) \quad , \tag{1.5}$$

$$\sum_{n=0}^{\infty} PDO_t(48n+22)q^n \equiv 8f_1f_2^5(mod\ 16) \quad , \tag{1.6}$$

$$\sum_{n=0}^{\infty} PDO_t(48n+14)q^n \equiv 8f_1^3f_2^2a(q^2)(mod\ 16) \quad , \tag{1.7}$$

$$PDO_t(48n+30) \equiv 0(mod\ 16) \quad , \tag{1.8}$$

$$\sum_{n=0}^{\infty} PDO_t(48n+46)q^n \equiv 8f_1^3f_2f_6^3(mod\ 16) \quad . \tag{1.9}$$

where

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right).$$

2. Proof of Theorem 1.1.

From [8, Eq. (4.4)], we get

$$\sum_{n\geq 0} PDO_t(8n)q^n = 36q\frac{f_2^8f_3^7}{f_1^{13}}$$

Also, in [9, Eq. (1.1)], we prove that

$$\begin{aligned}\sum_{n=0}^{\infty} PDO_t(16n)q^n &\equiv 36\frac{f_2^6f_3^6}{f_1^9f_6^2}(mod\ 72)\\ &\equiv 36\frac{f_2^4f_3^6}{f_1^8f_6^2}.\frac{f_2^2}{f_1}(mod\ 72)\\ &\equiv 36\frac{f_2^4f_6^3}{f_2^4f_6^2}.\frac{f_2^2}{f_1}(mod\ 72)\\ &\equiv 36f_6.\frac{f_2^2}{f_1}(mod\ 72)\\ &\equiv 36f_6\big(f(q^3,q^6)+q\psi(q^9)\big)(mod\ 72).\end{aligned}$$

Thus, we have

$$\sum_{n=0}^{\infty} PDO_t\big(16(3n+1)\big)q^n \equiv 36f_2\psi(q^3)(mod\ 72),$$

which implies that

$$\sum_{n=0}^{\infty} PDO_t(48n+16)q^n \equiv 36f_2\psi(q^3)(mod\ 72).$$

This completes the proof of (1.1).

Also,

$$\sum_{n=0}^{\infty} PDO_t\big(16(3n)\big)q^n \equiv 36f_2f(q,q^2)\ (mod\ 72),$$

which implies that

$\sum_{n=0}^{\infty} PDO_t(48n)q^n \equiv 36 f_2 f(q,q^2)\ (mod\ 72)$.

This completes the proof of (1.2).

and

$PDO_t(16(3n+2)) \equiv 0 (mod\ 72)$,

which implies that

$PDO_t(48n+32) \equiv 0 (mod\ 72)$.

This completes the proof of (1.3).

Thus, we complete the proof of the theorem.

3. Proof of Theorem 1.2.

Now, from [7, Eq. (1.7)], we have

$$\sum_{n=0}^{\infty} PDO_t(8n+6)q^n = 8\left(2\frac{f_2^{16} f_6^{10}}{f_1^{17} f_3^3 f_{12}^4} - q\frac{f_2^{28} f_3 f_{12}^4}{f_1^{21} f_6^2 f_4^8} - 16q^2\frac{f_2^4 f_3 f_4^8 f_{12}^4}{f_1^{13} f_6^2}\right)$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} PDO_t(8n+6)q^n &\equiv 8q\frac{f_2^{28} f_3 f_{12}^4}{f_1^{21} f_6^2 f_4^8} (mod\ 16)\\ &\equiv 8q\frac{f_2^{28} f_{12}^4}{f_1^{18} f_6^2 f_4^8}.\frac{f_3}{f_1^3} (mod\ 16)\\ &\equiv 8q\frac{f_2^{28} f_{12}^4}{f_2^9 f_{12} f_2^{16}}.\frac{f_3}{f_1^3} (mod\ 16)\\ &\equiv 8q f_2^3 f_{12}^3\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right)(mod\ 16),\end{aligned}$$

From which we extract

$$\begin{aligned}\sum_{n=0}^{\infty} PDO_t(16n+6)q^n &\equiv 8q f_1^3 f_6^3\frac{f_2^2 f_3 f_6^2}{f_1^7} (mod\ 16)\\ &\equiv 8q\frac{f_2^2 f_3 f_6^5}{f_1^4} (mod\ 16)\\ &\equiv 8q f_3 f_6^5 (mod\ 16),\end{aligned}$$

From which we further extract that

$PDO_t(16(3n)+6) \equiv 0 (mod\ 16)$,

which implies that

$PDO_t(48n+6) \equiv 0 (mod\ 16)$.

This completes the proof of (1.4).

Also,

$PDO_t(16(3n+2)+6) \equiv 0 (mod\ 16)$,

which implies that

$PDO_t(48n+38) \equiv 0 (mod\ 16)$.

This completes the proof of (1.5).

and

$\sum_{n=0}^{\infty} PDO_t(16(3n+1)+6)q^n \equiv 8 f_1 f_2^5 (mod\ 16)$,

which implies that

$\sum_{n=0}^{\infty} PDO_t(48n+22)q^n \equiv 8 f_1 f_2^5 (mod\ 16)$.

This completes the proof of (1.6).

And,

$$\begin{aligned}\sum_{n=0}^{\infty} PDO_t\ (8(2n+1)+6)q^n &\equiv 8 f_1^3 f_6^3\frac{f_2^6 f_3^3}{f_1^9 f_6^2} (mod\ 16)\\ &\equiv 8\frac{f_2^6 f_3^3 f_6}{f_1^6} (mod\ 16)\\ &\equiv 8\frac{f_2^6 f_3^3 f_6}{f_2^3} (mod\ 16)\\ &\equiv 8 f_2^3 f_3^3 f_6 (mod\ 16).\end{aligned}$$

Thus,

$\sum_{n=0}^{\infty} PDO_t(16n+14)\, q^n \equiv 8f_3^3 f_6\{f_6 a(q^6) - 3q^2 f_{18}^3\}(mod\ 16)$,
From which we extract
$\sum_{n=0}^{\infty} PDO_t(16(3n)+14)\, q^n \equiv 8f_1^3 f_2^2 a(q^2)(mod\ 16)$,
which implies that
$\sum_{n=0}^{\infty} PDO_t(48n+14)\, q^n \equiv 8f_1^3 f_2^2 a(q^2)(mod\ 16)$.
This completes the proof of (1.7).
Also,
$\sum_{n=0}^{\infty} PDO_t(16(3n+1)+14)\, q^n \equiv 0(mod\ 16)$,
which implies that
$PDO_t(48n+30) \equiv 0(mod\ 16)$.
This completes the proof of (1.8).
and,
$\sum_{n=0}^{\infty} PDO_t\ (16(3n+2)+14)q^n \equiv 8f_1^3 f_2 f_6^3(mod\ 16)$,
which implies that
$\sum_{n=0}^{\infty} PDO_t\ (48n+46)q^n \equiv 8f_1^3 f_2 f_6^3(mod\ 16)$.
This completes the proof of (1.9).
Thus, we complete the proof of the theorem.

3. References

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