Hopf-Bifurcation Induced by Delay Parameter in Depletion of Forest Biomass due to Wood-based Industries

Pankaj Kumar[1]

pankaj.kumar1@lpu.co.in
[1]School of Chemical Engineering and Physical Sciences
Lovely Professional University, Phagwara, Punjab, India.

Abstract: In this article, the effect of delay parameter on the wood-based industries depleting the forest biomass is analysed using a mathematical model. The logistic growth of forest biomass is assumed in the lack of wood-based industries. System behavior within all feasible equilibria is studied using differential equation stability theory. The delay parameter τ is a key parameter in the proposed model, and the system shows complex behaviour by exhibiting Hopf-bifurcation about interior equilibrium. The calculated procedural results are substantiated using simulation with MATLAB.
Key words: Forest biomass, Wood industry, Time delay, Hopf bifurcation.

Introduction

Mathematical modelling is an alternative method for analysing nutrient-based population dynamics under the influence of several realities, such as toxicants, diseases, etc. [1-6]. Until the 20th century, the majority of North American transportation systems were made of wood. Forests has been the precious gift of god to mankind right from the beginning of human civilization including food, shelter, medicine etc. Forests are required for medicine, bringing rain, protecting us from floods and what not[7-9]. Doon Valley in northern India is the typical example of depletion of forest biomass under the adverse process of limestone quarries, wood and paper-based industries. Forests have a pivotal role in protecting the climate and in providing people with the necessary requirements [10-14]. Yet forests suffer from depletion [1,3,4,14] due to greenhouse effect and the transformation of forest lands into non-forest. Manisha et al [2] gave a mathematical model suggesting how the use of synthetic based industries can reduce the depletion of forest biomass due to wood-based industries. Kaur and Preet [9] undertook a study to analyse the situation of fertility rate in the districts of Punjab. It has shown that fertility rate has decreased in leader districts. Sharma et al [12] used plastic track detectors for the measurement of concentration of toxic metals in soil samples collected from some villages of Kangra district, Himachal Pradesh, India. Kalra and Kumar [5,6,7,8] studied the impact of delay parameter in plant growth dynamics effected by toxicant using different models. Ruan and Wei [10] analysed the nature of zeros of exponential characteristic equation. The stability analysis of equilibrium points involving a non-linear system of delay differential equations is carried out by Ruan [11, 12].

In the view of above, therefore, in this paper the problem related role of delay in stability and bifurcation analysis of exhaustion of forest biomass due to wood-based industries is studied using the following mathematical model

Proposed Model

The dynamics of depletion of forest biomass due to wood industries is represented by succeeding framework of equations including two variables: density of forest biomass F_B and density of wood-based industries W_I.

$\frac{dF_B}{dt} = gF_B\left(1 - \frac{F_B}{C}\right) - \alpha_1 F_B(t-\tau)W_I$ (1)

$\frac{dW_I}{dt} = \beta_1 F_B W_I - \gamma_1 W_I$ (2)

Where: $F_B(0) > 0, W_I(0) > 0$ for all t and $F_B(t-\tau) =$ constant for $t \in [0,\tau]$.

The variables parameters and considered in this model are interpreted in Table 1:

Table 1: Description of the variables and parameters of the system (1)- (2)

Variables/Parameter	Description
$F_B(t)$	Density of forest biomass growing logistically
$W_I(t)$	Density of wood-based industries
g	Inherent growth rate of forest population
C	Carrying capacity
α_1	Exhaustion rate of forest biomass
β_1	Growth rate of wood-based industries
γ_1	Exhaustion rate of wood industry
τ	Delay Parameter

For non-denationalization, let $b = \frac{F_B}{C}$ and $w = \frac{W_I}{C}$, The re-scaled system becomes:

$\frac{db}{dt} = b(1-b) - \alpha b(t-\tau)w$ (3)

$\frac{dw}{dt} = \beta bw - \gamma w$ (4)

Where $\alpha = \frac{\alpha_1 C}{g}, \beta = \frac{\beta_1 C}{g}, \gamma = \frac{\gamma_1}{g}$ and $b(0) > 0, w(0) > 0, b(t-\tau) =$ constant for $t \in [0,\tau]$.

Analysis of the Model

Boundedness:

From equation (3):

$\frac{db}{dt} = b(1-b) - \alpha b(t-\tau)w$

$\Rightarrow \frac{db}{dt} \le b(1-b) \Rightarrow \lim_{t\to\infty} b \le 1$. So, $b_u = 1$

Also $\frac{db}{dt} \ge b(1-b) - \alpha b w_u \Rightarrow \frac{db}{dt} \ge b(1-\alpha b w_u) - b^2$

Suppose $(1-\alpha b w_u) = c_1 \Rightarrow \lim_{t\to\infty} b \ge c_1$. So, $b_l = (1-\alpha b w_u)$

From equation (3) and (4):

$\beta\frac{db}{dt} + \alpha\frac{dw}{dt} = \beta b - \beta b^2 - \alpha\gamma w$

$\Rightarrow \frac{d}{dt}(\beta b + \alpha w) \le \beta b - \alpha\gamma w \le \beta b - \varphi\beta b + \varphi\beta b - \alpha\gamma w$

$\le \varphi\beta b_u - \beta b(\varphi - 1) - \alpha\gamma w \le \varphi\beta b_u - m\beta b - \alpha\gamma w$

Let $m_1 = \min(m,\gamma)$, we get:

$\frac{d}{dt}(\beta b + \alpha w) \le \varphi\beta - m_1(\beta b + \alpha w) \Rightarrow \lim_{t\to\infty}(\beta b + \alpha w) \le \frac{\varphi\beta}{m_1}$

$\Rightarrow \alpha w_u = \frac{\varphi\beta}{m_1}$ So, $w_u = \frac{\varphi\beta}{\alpha m_1}$

Hence, all the solutions of the system of equations (1)- (2) lie in the two-dimensional region $\mathcal{R} = \left\{(b,w) \in R^{2+}: 0 < b_l \le b \le b_u, 0 < w \le w_u, b_l = 1 - \frac{\varphi\beta}{m_1}, b_u = 1, w_u = \frac{\varphi\beta}{\alpha m_1}\right\}, t \to \infty$, where $m_1 = \min(m,\gamma)$ with initial conditions $b \ge 0, w \ge 0, \forall\, t > 0$ and $b(t-\tau) =$ constant for $t \in [0,\tau]$.

Positivity of Solutions:

From equation (4): $\frac{dw}{dt} \ge -\gamma w \Rightarrow \frac{dw}{w} \ge -\gamma dt \Rightarrow w \ge e^{-\gamma t}$

From equation (3): $\frac{db}{dt} \ge -b^2 - \alpha bw \Rightarrow \frac{db}{dt} \ge -b(1+\alpha w) \Rightarrow \frac{db}{dt} \ge -b\left(1 + \frac{\varphi\beta}{m_1}\right)$

$\Rightarrow \frac{db}{b} \geq -\left(1+\frac{\varphi\beta}{m_1}\right)dt \Rightarrow b \geq e^{-\left(1+\frac{\varphi\beta}{m_1}\right)t}$

So, $b \geq 0, w \geq 0 \ \forall\, t.$

Hence solution set of the system of equations stay positive for all t. It ensures that system persists.

Equilibrium points:

There are three equilibrium points $E_1(b^*=1, w^*=0), E_2(b^*=0, w^*=0)$ and $E_3(b^* \neq 0, w^* \neq 0)$:

The third equilibrium $E_3(b^* \neq 0, w^* \neq 0)$ is found by annihilating two derivatives $\frac{db^*}{dt}$ and $\frac{dw^*}{dt}$. i.e. $\frac{db^*}{dt}=0$ and $\frac{dw^*}{dt}=0$

From $\frac{dw^*}{dt}=0$; we get the first isocline as:

$$b^*=\frac{\gamma}{\beta} \quad (5)$$

From $\frac{dF^*}{dt}=0$; we get the second isocline as:

$$w^*=\frac{\beta-\gamma}{\alpha\beta} \quad (6)$$

The intersection of these two isoclines give the equilibrium E^*.It is assumed that at the point of equilibriums: $b^*(t-\tau) \cong b^*(t)$.

Stability of Equilibrium $E^*$and Hopf bifurcation:

The system of equations governing the nutrient-plant biomass mechanism at the equilibrium $E_3(b^* \neq 0, w^* \neq 0)$ is given by:

$$\frac{db^*}{dt}=b^*(1-b^*)-\alpha b^*(t-\tau)w^* \quad (7)$$

$$\frac{dw^*}{dt}=\beta b^* w^*-\gamma w^* \quad (8)$$

The characteristic equation associated with the system of equations (7)-(8) is given by:

$$\lambda^2+p\lambda+q+(r\lambda+s)e^{-\lambda\tau}=0 \quad (9)$$

Where $p=(2b^*+\gamma)-(1+\beta b^*), q=(\beta b^*+2b^*\gamma)-(\gamma+2\beta b^*), r=\alpha w^*$, $s=\alpha w^*(\gamma-\beta b^*)$

$p \geq 0$ if $(2b^*+\gamma) \geq (1+\beta b^*)$, $q \geq 0$ if $(\beta b^*+2b^*\gamma) \geq (\gamma+2\beta b^*)$, $r \geq 0$ if $\alpha w^* \geq 0$, $s \geq 0$ if $\gamma \geq \beta b^*$

When $\tau=0$, the equation (9) becomes:

$$\lambda^2+(p+r)\lambda+(q+s)=0 \quad (10)$$

By Routh-Hurwitz's criteria, roots of equation (10) will have negative real part i.e. the system is stable if:

$(\boldsymbol{\mathcal{H}_1})$: $(p+r)>0$;

$(\boldsymbol{\mathcal{H}_2})$: $(q+s)>0$

Now, we would like to check the shifting of negative real part of the roots to positive real parts with variations in the values of τ.

Let $\lambda=i\omega$ be a root of equation (9), then equation (9) becomes:

$$(i\omega)^2+p(i\omega)+q+(r(i\omega)+s)e^{-(i\omega)\tau}=0$$

$$\Rightarrow -\omega^2+p(i\omega)+q+(r(i\omega)+s)(\cos\omega\tau-i\sin\omega\tau)=0$$

Separating real and imaginary parts:

$$-\omega^2+q=-s\cos\omega\tau-r\omega\sin\omega\tau \quad (11)$$

$$p\omega=-r\cos\omega\tau+s\sin\omega\tau \quad (12)$$

It follows that ω satisfies:

$$\omega^4-(r^2-p^2+2q)\omega^2+(q^2-s^2)=0 \quad (13)$$

The two roots of equation (13) are:

$$\omega_{1,2}^2 = \frac{(r^2-p^2+2q)\pm\sqrt{(r^2-p^2+2q)^2-4(q^2-s^2)}}{2} \qquad (14)$$

None of the two roots $\omega_{1,2}^2$ is positive if:

$(\boldsymbol{\mathcal{H}_3})$: $(r^2 - p^2 + 2q) < 0$ *and* $(q^2 - s^2) > 0$ *or* $(r^2 - p^2 + 2q)^2 < 4(q^2 - s^2)$

That means equation (14) does not have positive roots if condition $(\mathcal{H}_3)$ holds.

We have the following Conjecture (Ruan [12])

Conjecture 1. If $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold, then all the roots of equation (9) have negative real parts for all $\tau \geq 0$.

On the other hand, if:

$(\boldsymbol{\mathcal{H}_4})$: $(q^2 - s^2) < 0$ *or* $(r^2 - p^2 + 2q) > 0$ *and* $(r^2 - p^2 + 2q)^2 = 4(q^2 - s^2)$

Then, +ve root of equation (11) is ω_1^2.

On the same basis, if:

$(\boldsymbol{\mathcal{H}_5})$: $(q^2 - s^2) > 0$ *or* $(r^2 - p^2 + 2q) > 0$ *and* $(r^2 - p^2 + 2q)^2 > 4(q^2 - s^2)$

Then, two +ve roots of equation (11) are $\omega_{1,2}^2$.

In both- $(\mathcal{H}_4)$and $(\mathcal{H}_5)$, the equation (9) has purely imaginary roots when τ takes certain values. The critical values $\tau_j^{\pm}$ of τ can be calculated from the system of equations (9)-(10), given by:

$$\tau_j^{\pm} = \frac{1}{\omega_{1,2}} cos^{-1}\left[\frac{s(\omega_{1,2}^2 - q) - pr\omega_{1,2}^2}{r^2\omega_{1,2}^2 + s^2}\right] + \frac{2j\pi}{\omega_{1,2}}, j = 0,1,2, \ldots \qquad (15)$$

The above discussion can be condensed in succeeding conjecture (Ruan [12])

Conjecture 2. (I) If $(\mathcal{H}_1) - (\mathcal{H}_2)$ *and* $(\mathcal{H}_4)$ hold and $\tau = \tau_j^+$, then equation (9) has a pair of purely imaginary roots $\pm i\omega_1$.

(II) If $(\mathcal{H}_1) - (\mathcal{H}_2)$ *and* $(\mathcal{H}_5)$ hold and $\tau = \tau_j^+\left(\tau = \tau_j^- \; respectively\right)$, then equation (9) has a pair of purely imaginary roots $\pm i\omega_1(\pm i\omega_2 \; respectively)$.

Our expectation is the shifting of negative real part of some roots of equation (9) to positive real part when $\tau > \tau_j^+$ and $\tau < \tau_j^-$. To look into this possibility, let us denote:

$\tau_j^{\pm} = \mu_j^{\pm}(\tau) + i\omega_j^{\pm}(\tau); j = 0,1,2,3, \ldots$

The roots of equation (9) satisfy: $\mu_j^{\pm}\left(\tau_j^{\pm}\right) = 0, \omega_j^{\pm}\left(\tau_j^{\pm}\right) = \omega_{1,2}$

We can verify that the following transversality condition holds:

$\frac{d}{d\tau}\left(Re\; \lambda_j^+\left(\tau_j^+\right)\right) > 0 \; and \; \frac{d}{d\tau}\left(Re\; \lambda_j^-\left(\tau_j^-\right)\right) < 0$

It concludes that $\tau_j^{\pm}$ are bifurcating values. The succeeding postulate gives the scattering of the zeros of the equation (9) (Ruan [12])

Postulate: Let $\tau_j^+ (j = 0,1,2,3, \ldots)$ be defined by equation (15).

(I) If $(\mathcal{H}_1), (\mathcal{H}_2)$ hold, then all the roots of equation (9) have -ve real parts for all $\tau \geq 0$.

(II) If $(\mathcal{H}_1), (\mathcal{H}_2)$ *and* $(\mathcal{H}_4)$ hold and when $\tau \in [0, \tau_0^+)$, then all the roots of equation (9) have -ve real parts. When $\tau = \tau_0^+$, then equation (9) has a pair of purely imaginary roots $\pm i\omega_1$. When $\tau > \tau_0^+$, equation (7) has at least one root with +ve real part.

(III) If $(\mathcal{H}_1), (\mathcal{H}_2)$ *and* $(\mathcal{H}_5)$ hold, then there is a +ve integer m such that $0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- - -- < \tau_{m-1}^- < \tau_m^+$ and there are m switches from stability to instability. This means, when $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), ---, (\tau_{m-1}^-, \tau_m^+)$, all the roots of equation (9) have negative real parts. When $\tau \in (\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), ---, (\tau_{m-1}^+, \tau_{m-1}^-)$ and $\tau > \tau_m^+$, equation (9) has at least one root with +ve real part.

Numerical Example

The following set of parametric values is taken to represent graphically the dynamics depicted by the system of equations (1)- (2)

$\alpha = 0.6, \beta = 0.6, \gamma = 0.4$ with: $b(0) = 0.5, w(0) = 0.5$

The change of behaviour of the system of equations (1)-(2) from being stable to complex dynamics about the equilibrium $E_3(0.6667,0.8889)$ for different values of delay parameter τ is shown below:

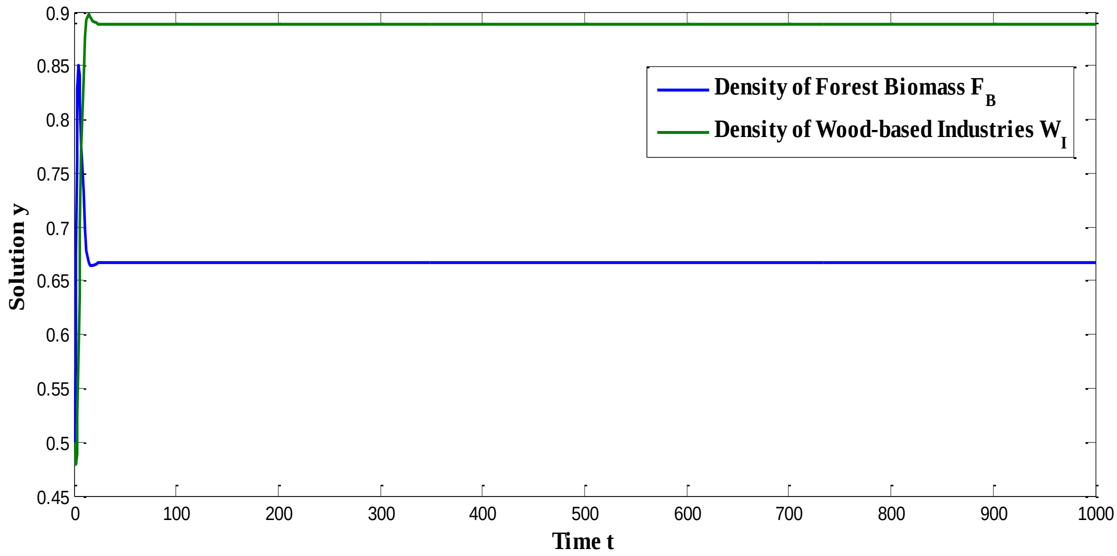

Figure 1. The equilibrium $E_3(0.6667,0.8889)$ is stable in the absence of delay i.e. $\tau = 0$

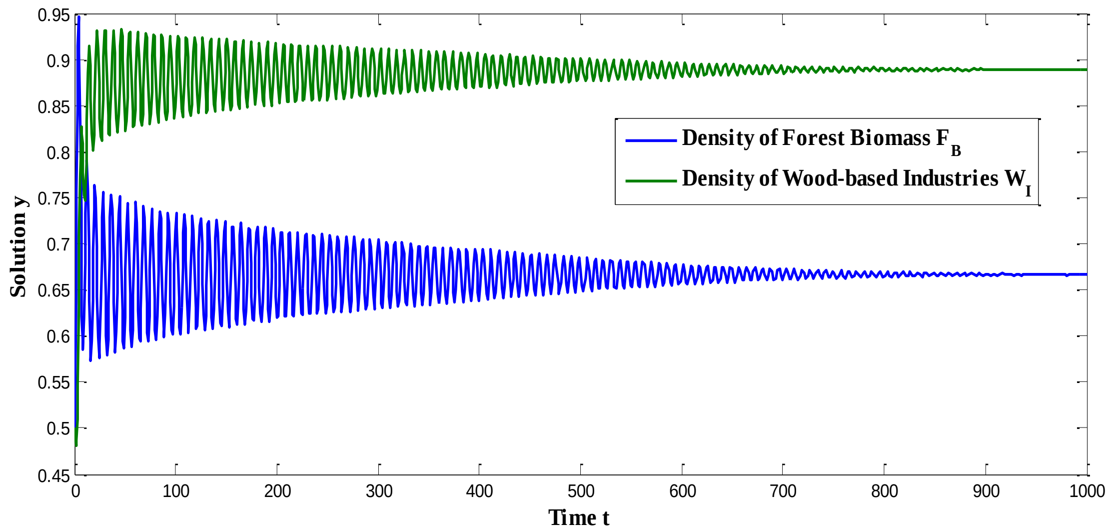

Figure 2. The equilibrium $E_3(0.6667,0.8889)$ is asymptotically stable when delay is below the critical point i.e. $\tau < 2.37$

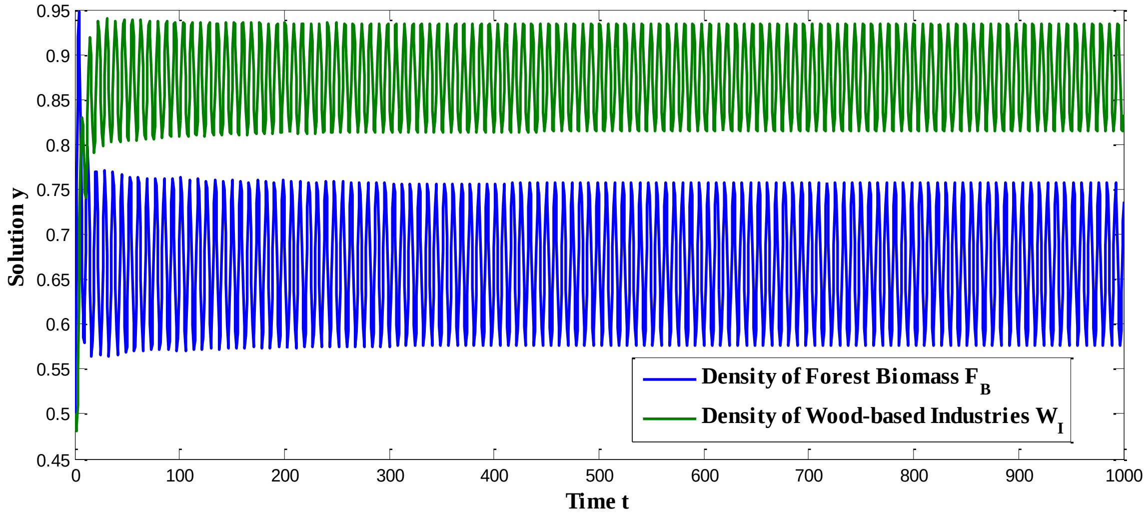

Figure 3. The equilibrium $E_3(0.6667,0.8889)$ shows Hopf bifurcation when the delay passes through the critical point i.e. $\tau \geq 2.37$

Conclusion

The role of delay on nutrient-plant biomass mechanism is studied with the help of proposed model. The state variables considered are: density of forest biomass F_B and density of wood industries W_I. The boundedness of the system is proved using usual comparison theorem. Positivity of the solutions shows that both the variables considered being real in natural phenomenon always remain positive at all the times. In the absence of delay, the equilibrium E_3 is absolutely stable as shown in figure 1. The same fact is also supplemented by $(\mathcal{H}_1) - (\mathcal{H}_2)$ as in conjecture 1. When the value of delay parameter τ is below the critical point i.e. $\tau < 2.37$, the equilibrium starts losing stability and leads to asymptotical stability as shown in figure 2. The moment, the delay parameter τ crosses the critical value i.e. $\tau \geq 2.37$, the equilibrium exhibits the complex dynamics in the form of Hopf bifurcation. This observation of complex behaviour shown by the system (1)-(2) as shown by figure 3 which is in agreement with $(\mathcal{H}_4) - (\mathcal{H}_5)$ as in conjecture 2.

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