Some Special Properties Of Ideals And Congruences In Lattice Ordered Commutative Loops

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Abstract:
This manuscript illustrates the significance of a normal subloop, l-morphism, l-ideal of an l-loop also we have succeeded in determining a corresponding congruence relation on the l-loop and establishing a one-to-one correspondence between the l-ideals and congruence relations of an l-loop A.

Key words: Loops, partial order, lattices, ordered abelian groups, Ideals-congruence relations

A.M.S. Mathematics subject classification numbers (2020): 20N05, 06A06, 06Bxx, 06F20, 06B10

1. INTRODUCTION:

In 1967, G.Birkhoff in Lattice Theory, various properties of lattice ordered groups were established. In 1970, T.Evans described about lattice ordered loops and quasigroups. In 1990, Hala made a description on quasigroups and loops [1-3]. In view of this a lot of interest has been shown different authors develop these concepts in different algebraic systems. In 2014, V. B. V. N. Prasad and J. VenkateswaraRao, were gave Categorization of Normal Sub Loop and Ideal of Loops. In 2019, B.Sailaja, V.B.V.N.Prasad, developed exploring the axiom of excluded middle and axiom of contradiction in fuzzy sets. In 2020, R.Sunil Kumar and V.B.V.N.Prasad were giving some special characteristics of Atoms in Lattice ordered loops and in 2020, V.B.V.N.Prasad, T.Rama Rao and some authors were gave Some Basic Principles on Posets, Hasse diagrams and lattices. In 2020, V. B. V. N. Prasad1, K. Prasad, Mudda Ramesh, Rama Devi Burri and T. Rama Rao were established Some Special Characteristics of Lattice Ordered Commutative Loops. In 2020,Praveen VardhanKuppili, V.B.V.N. Prasad, Applicationof Translatesof Vague Setson Suspected CasesofCorana Virus Disease-2019 (COVID-19), [4-9].

2. L-IDEALS:

Definition 2.1: Let A be an l-loop. For a, b∈A, we define $a*b= (a-b)\vee(b-a) =a\vee b-a\wedge b$.

Lemma 2.1: Let A be a l-loop. Then $\forall a,b \in A$

1 $a*b \geq 0$ with equality if and only if a=b.
2 $a*b=b*a$.
3. $(a\vee b) *(a\wedge b) =a*b$.

Proof: The proof follows easily.

Definition 2.2: A sub loop R of a loop A is called a normal sub loop of A [2], if and only if
$(R+x)+y=R+(x+y), \forall x,y \in A$.
It is not hard to see that a normal sub loop R of a loop A partitions it.

Definition 2.3: A nonempty subset R of an l-loop A is called a l-ideal if and only if R is a normal sub loop of A in which $a\in R, b\in A, b*0\leq a*0 \Longrightarrow b\in R$.

Theorem 2.1: A nonempty subset R of a l-loop A is an l-ideal if and only if R is a convex normal l-sub loop of A.

Proof: Let R be an l-ideal and $a\leq x\leq b$, a, $b\in R$.
Then $a\wedge b\leq x\leq a\vee b$.

Now $x*0\leq(a\vee b)\vee(0-a\wedge b)$
$=(a\vee b)\vee(0-a)\vee(0-b)=(a*0)\vee(b*0)\leq a*0+b*0$.
$\Longrightarrow x\in R$.

Hence R is convex.
Also $(a\vee b)*0=(a\vee b)\vee(0-a\vee b)\leq a*0\vee b*0$
$\leq a*0+b*0$
$\Longrightarrow a\vee b\in R$,
and $(a\wedge b)*0=(a\wedge b)\vee[0-(a\wedge b)]\leq(a*0)\vee(b*0)\leq(a*0)+(b*0)$
$\Longrightarrow a\wedge b\in R$.

Thus, R is a convex sublattice of A.

Conversely let R be a convex normal l-subloop of A.

Let $a\in R, b*0\leq a*0$.
Now $b\leq b*0\leq a*0$
$\Longrightarrow 0-b\geq 0-(a*0)$ so that $0-b\in R$ by conversity in R and hence $b=0-(0-b)\in R$.
This completes the proof.

Definition 2.4: An l-morphism of an l-loop A into a l-loop B is a mapping $f: A\longrightarrow B$ such that $\forall a,b\in A$,
1. $f(a+b)=f(a)+f(b)$,
2. $f(a-b)=f(a)-f(b)$,
3. $f(a\vee b)=f(a)\vee f(b)$,
4. $f(a\wedge b)=f(a)\wedge f(b)$.

Corollary 2.1: If $f:A\longrightarrow B$ is an l-morphism of an l-loop A into an l-loop B, then $\forall a,b\in A$, $f(a*b)=f(a)*f(b)$.

Definition 2.5: An equivalence relation θ on an l-loop A is a congruence relation if for $a,b\in A, a\equiv b(\theta)\Longrightarrow a+x\equiv b+x(\theta), a-x\equiv b-x(\theta), x-a\equiv x-b(\theta), a\vee x\equiv b\vee x(\theta)$ and $a\wedge x\equiv b\wedge x(\theta), \forall x\in A$.

Since an l-loop A is equationally definable, we have the following:

Theorem 2.2: Let θ be a congruence relation on an l-loop A. For any $a\theta, b\theta \in A/\theta$, define $a\theta+b\theta=(a+b)\theta, a\theta-b\theta=(a-b)\theta$ and $a\theta$ is positive iff $a\theta$ contains a positive element. Then $(A/\theta,+,-,\leq)$ is an l-loop.

The following theorem establishes a one-one correspondence between the l-ideals and the congruence relations of an l-loop.

Theorem 2.3: There is one-to-one correspondence between the l-ideals and congruence relations of an l-loop A.

Proof: Let R be an l-ideal ofA. Define

(a) a≡b(θ_R) iff a=b or a,b∈R.

Clearly $\theta_R$is an equivalence relation on A.

Next define a relation θ'_Ras follows:

(b) a≡b(θ'_R) iff ∃ elements t,t',c,s,s'∈ A such that a=[t∨(c+s)]∧t' and b=[t∨(c+s')]∧t' where s≡s'(θ_R).

We know show that θ'_R is reflexive,symmetric and satisfies the substitutionproperty for +,-,∨,∧.

1. Sincea=[(a∧b) ∨ (0+a)]∧(a∨b), it follows that a≡a (θ'_R).
2. Clearly θ'_R is symmetric.
3. Let a≡b (θ'_R). Then ∃ elements t,t',c,s,s'∈A such that a=[t∨(c+s)]∧t'

and b=[t∨(c+s')]∧t' where s≡s'(θ_R).
Now x+a = x+t'∧ [t∨(c+s)]
= (x+t') ∧ [(t+x)∨{x+(c+s)}]
= (x+t') ∧ [(t+x)∨{(x+c)+s_1}] for some s_1∈ R.
Similarly,x+b=(x+t')∧[(t+x)∨{(x+c) +s_1'}] where s_1'∈ R.

Thus x+a≡x+b (θ'_R).

In our subsequent paper of ideal theory,we have simplified the proof of this theorem. If S is an l-ideal of A then the a≡b(θ_s) ⟺ a-b∈S gives congruence relation.
Again
a-x = t'∧[t∨(c+s)]-x
=(t'-x)∧[(t-x)∨{(c+s)-x}]
=(t'-x)∧[(t-x)∨{(c-s) +s_2}] for some s_2∈ R.
Similarly, b-x=(t'-x)∧[(t-x)∨{(c-x) +s_2'}] where s_2'∈ R.
Thus a-x≡b-x (θ'_R).
Now
x-a = x-[t'∨{t∨(c+s)}]
=[x-(t'∧t)]∧[(x-t')∨{(x-(c+s)}]
=[x-(t'∧t)]∧[(x-t')∨{(x-c) +s_3}] for some s_3∈R
Similarly, x-b=[(x-(t'∧t)]∧[(x-t')∨{(x-c) +s_3'}] where s_3'∈ R

Thus x-a≡x-b ($\square'_{\square}$).
Since A is a distributive lattice it is easy to show that (x∨a)≡x∨b ($\square'_{\square}$) andalso clearly x∧a≡x∧b($\square'_{\square}$).

Thus $\square'_{\square}$satisfies substitution property for +,-, ∨, ∧.
Now let $\square''_{\square}$be the transitive extension of $\square'_{\square}$.

It is easy to see that $\square_R \le \square'_{\square} \le \square''_{\square}$

Hence an l-ideal R inA defines a congruence relation $\square''_{\square}$ in A.
Conversely let □ be a congruence relation on A and R be the set of all x≡(0)□

Clearly a,b ∈ R ⟹ a+b ∈ R.
Let x∈R,y*0≤x*0.
Now x≡0(□) ⟹ x*0≡0(□)⟹ (y*0)∨(x*0)≡0(□)
⟹y*0≡0(□) ⟹ y∨(0-y) ≡ 0(□) ⟹ y∧0≡y≡y∨0(□)
⟹y∨0≡(y∧0)∨0=0(□) ⟹ □ ≡ 0(□) ⟹ y∈R.

Hence R is an l-ideal.
Now let $\square''_{\square}$be the congruence relation defined by R.
Then a≡b ($\square''_{\square}$) ⟹ ∃ a sequence a=z_0,z_1,z_2,..z_n = b in A such that z_i ≡ z_{i+1}($\square'_{\square}$).
Then z_i=[t∨(c+s)]∧t',

Z_{i+1}=[t∨(c+s')]∧t' where s≡s'($\square_R$) that is s, s'∈ R=0[□].
Since □ is a congruence relation and s≡s' (□),z_i≡z_{i+1}(□) and by transitivity of □, it follows that a≡b (□) showing $\square''_{\square}$ ⊆ □.

Again a ≡ b(□) ⟹ a −□ ≡ 0(□) ⟹ a-b∈ R.
⟹a-b≡ 0($\square_R$) ⟹a ≡ b($\square_R$) ⟹ a≡b ($\square''_{\square}$) ⟹ □ ≤ $\square''_{\square}$.
$\square''_{\square}$= □. This completes the proof

Corollary 2.2: The congruence relation on any l-loop A are the partitions of A into the cosets of its different l-ideals.

Theorem 2.4: The congruence relations on any l-loop A is a complete Algebraic Brouwerian lattice.

Proof: Follows in the same lines as in l-groups[l].

Corollary 2.3: The l-ideals of an l-loop A forms a complete Algebraic Brouwerian lattice.

5. CONCLUSION:

This research work initiates some important aspects in lattice ordered algebraic structures and especially some important properties of l-ideals and congruences in lattice ordered loops were established. Further there is so much scope for the remaining algebraic structures in lattice ordered loops.

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