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MATRIX REPRESENTATION OF BIVARIATE BI-PERIODIC JACOBSTHAL POLYNOMIALS

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Abstract - In this paper we will introduce Bi-variate Bi- periodic Jacobsthal matrix polynomial sequence which is defined as $\hat{J}_{n}(x,z) = u(x,z)\hat{J}_{n-1}(x,z) + 2h(z)\hat{J}_{n-2}(x,z)$ ***if*** n ***is even and*** $\hat{J}_{n}(x,z) = v(x,z)\hat{J}_{n-1}(x,z) + 2h(z)\hat{J}_{n-2}(x,z)$ ***if*** n ***is odd with initial conditions*** $\hat{J}_{0}(x,z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ***and*** $\hat{J}_{1}(x,z) = \begin{bmatrix} v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} \\ 1 & 0 \end{bmatrix}$***. We will find generating function, well-known Cassini's identity, determinant and Binet's formula for the polynomial sequences. Also, we will find summation formula for this generalized matrix polynomial sequence.*** $\hat{H}_{q}(x,z)$ ***generating matrix and a few related results for Bi-variate Bi-periodic Jacobsthal polynomial sequence will also be discussed.***

Keywords: Bi-variate Bi-periodic Jacobsthal polynomials, Cassini's identity, Binet's formula, Generating Function, Generating Matrix

A.M.S. subject classification: 11B83

1. INTRODUCTION

In [1] Bala and Verma defined a polynomial sequence called Bi-periodic Jacobsthal polynomial sequence and discussed some results. Hadamard product for Bi-periodic Fibonacci numbers and Bi-periodic Lucas generating matrix sequences was defined by Coskun and Taskara [2]. They also discussed some properties. In [3] Coskun and Taskara introduced Matrix representation of Bi-periodic Fibonacci numbers. In [4] Uygun introduced new sequence called Bi-periodic Jacobsthal sequence and discussed their properties. In [5] Uygun introduced Matrix representation of Bi-periodic Jacobsthal polynomials and explained a few properties of the representation. In [6] Uygun defined Bivariate Jacobsthal and bivariate Jacobsthal-Lucas matrix polynomial sequences, established some relations and discussed several properties. Generalized bivariate Jacobsthal and Jacobsthal-Lucas polynomial sequences was defined by Uygun [7] and a few properties was also discussed by him. Verma and Bala [8] defined a polynomial sequence called Bi-variate Bi-periodic Fibonacci polynomial sequence and established some important results.

1.(a) Definitions and Lemmas

Definition 1.1: For any $u(x,z)$, $v(x,z)$ and $h(z)$ belonging to $\mathbb{R} - \{0\}$, x, z belonging to $\mathbb{R}$ and $n \in \mathbb{N}$ the bivariate bi-periodic Jacobsthal matrix polynomials is given

$$\hat{J}_{n}(x,z) = \begin{cases} u(x,z)\hat{J}_{n-1}(x,z) + 2h(z)\hat{J}_{n-2}(x,z) & if\ n\ is\ even \\ v(x,z)\hat{J}_{n-1}(x,z) + 2h(z)\hat{J}_{n-2}(x,z) & if\ n\ is\ odd \end{cases} \quad n \geq 2 \tag{1.1.1}$$

with initial conditions $\hat{J}_{0}(x,z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\hat{J}_{1}(x,z) = \begin{bmatrix} v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} \\ 1 & 0 \end{bmatrix}$.

Definition 1.2: Bi-variate Bi-periodic Jacobsthal $\hat{H}_{q}(x,z)$-generating matrix is defined as

$$\hat{H}_q(x,z)=\begin{bmatrix} v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} \\ 1 & 0 \end{bmatrix} \quad (1.1.2)$$

Definition 1.3: For any u (x, z), v(x, z) and $h(z)$ belonging to $\mathbb{R}-\{0\}$, x, z belonging to $\mathbb{R}$ and $n \in \mathbb{N}$ the bivariate bi-periodic Jacobsthal polynomials is given by

$$j_n(x,z) = \begin{cases} u(x,z)j_{n-1}(x,z) + 2h(z)j_{n-2}(x,z) & if\ n\ is\ even \\ v(x,z)j_{n-1}(x,z) + 2h(z)j_{n-2}(x,z) & if\ n\ is\ odd \end{cases} \quad n \geq 2 \quad (1.1.3)$$

with initial conditions $j_0(x,z) = 0, j_1(x,z) = 1$

$$j_0(x,z) = 0, \quad j_1(x,z) = 1, \quad j_2(x,z) = u(x,z), \quad j_3(x,z) = u(x,z)v(x,z) + 2h(z),$$
$$j_4(x,z) = u(x,z)^2 v(x,z) + 4u(x,z)h(z)$$

Characteristic equation of the bivariate bi-periodic Jacobsthal matrix polynomial and Bivariate Bi-periodic Jacobsthal polynomial

$$\lambda^2 - u(x,z)v(x,z)\lambda - 2u(x,z)v(x,z)h(z) = 0 \quad (1.1.4)$$

with roots $\lambda_1(x,z)$ and $\lambda_2(x,z)$ given by

$$\lambda_1(x,z) = \frac{u(x,z)v(x,z) + \sqrt{\left(u(x,z)v(x,z)\right)^2 + 8u(x,z)v(x,z)h(z)}}{2}$$

and

$$\lambda_2(x,z) = \frac{u(x,z)v(x,z) - \sqrt{(u(x,z)v(x,z))^2 + 8u(x,z)v(x,z)h(z)}}{2}$$

Lemma 1.4 The $\lambda_1(x,z)$ and $\lambda_2(x,z)$ defined by (1.1.3) satisfies the following properties

(i) $(\lambda_1 + 2h(z))\left(\lambda_2 + 2h(z)\right) = (2h(z))^2$
(ii) $\lambda_1 + \lambda_2 = u(x,z)v(x,z)$
(iii) $\lambda_1 \lambda_2 = -2u(x,z)v(x,z)h(z)$
(iv) $(\lambda_1)^2 + (\lambda_2)^2 = \left(u(x,z)v(x,z)\right)^2 + 4u(x,z)v(x,z)h(z)$
(v) $-(\lambda_1(\lambda_2 + 2h(z)) = 2h(z)\lambda_2$
(vi) $-(\lambda_2(\lambda_1 + 2h(z)) = 2h(z)\lambda_1$
(vii) $(\lambda_2 + 2h(z)) = \frac{(\lambda_2)^2}{u(x,z)v(x,z)}$
(viii) $(\lambda_1 + 2h(z)) = \frac{(\lambda_1)^2}{u(x,z)v(x,z)}$

We can use $\lambda_1(x,z) = \lambda_1$ and $\lambda_2(x,z) = \lambda_2$

Lemma 1.5: For any non-negative integer n, we have

$$\hat{J}_{2n}(x,z) = \left(u(x,z)v(x,z) + 4h(z)\right)\hat{J}_{2n-2}(x,z) - \left(2h(z)\right)^2 \hat{J}_{2n-4}(x,z) \quad (1.5.1)$$

$$\hat{J}_{2n+1}(x,z) = \left(u(x,z)v(x,z) + 4h(z)\right)\hat{J}_{2n-1}(x,z) - \left(2h(z)\right)^2 \hat{J}_{2n-3}(x,z)$$
(1.5.2)

***Proof*:** Proof is obvious by using definition of the Bivariate Bi-periodic Jacobsthal matrix polynomial sequence.

2. Main Results

***Theorem 2.1*:** For any n$\in \mathbb{N} \cup \{0\}$, the nth Bivariate Bi-periodic Jacobsthal matrix polynomial sequence is given by:

$$\hat{J}_n(x,z) = \begin{bmatrix} \left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(n)} j_{n+1}(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_n(x,z) \\ j_n(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(n)} j_{n-1}(x,z) \end{bmatrix}$$

Proof: We will proceed with help of Principle Mathematical Induction. Here j_n is denoted as bi-variate bi-periodic Jacobsthal polynomial. Firstly, we will check validity for *n=0* and *n=1* respectively as follows:

$$\hat{J}_0(x,z) = \begin{bmatrix} j_1(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_0(x,z) \\ j_0(x,z) & 2h(z) j_{-1}(x,z) \end{bmatrix}$$

Where $j_0(x,z) = 0,\ j_1(x,z) = 1,\ j_2(x,z) = u(x,z), and\ j_{-1}(x,z) = \frac{1}{2h(z)}$

So,

$$\hat{J}_0(x,z) = \begin{bmatrix} j_1(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_0(x,z) \\ j_0(x,z) & 2h(z) j_{-1}(x,z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{J}_1(x,z) = \begin{bmatrix} \left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(1)} j_2(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_1(x,z) \\ j_1(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(1)} j_0(x,z) \end{bmatrix} = \begin{bmatrix} v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} \\ 1 & 0 \end{bmatrix}$$

Now, let us suppose that the result is true for $n = k$

$$\hat{J}_k(x,z) = \begin{bmatrix} \left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k)} j_{k+1}(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_k(x,z) \\ j_k(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k)} j_{k-1}(x,z) \end{bmatrix}$$

Now, let us check the validity for $n = k + 1$. Let us assume that k is odd so $n = k + 1$ is even

$$\hat{J}_{k+1}(x,z) = \begin{cases} u(x,z)\hat{J}_k(x,z) + 2h(z)\hat{J}_{k-1}(x,z) & if\ k+1\ is\ even \\ v(x,z)\hat{J}_k(x,z) + 2h(z)\hat{J}_{k-1}(x,z) & if\ k+1\ is\ odd \end{cases}$$

$$\hat{J}_{k+1}(x,z) = u(x,z)^{\xi(k)} v(x,z)^{1-\xi(k)} \hat{J}_k(x,z) + 2h(z)\hat{J}_{k-1}(x,z)$$

$$\hat{J}_{k+1}(x,z) = u(x,z)^{\xi(k)} v(x,z)^{1-\xi(k)} \begin{bmatrix} \left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k)} j_{k+1}(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_k(x,z) \\ j_k(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k)} j_{k-1}(x,z) \end{bmatrix} +$$

$$2h(z)\begin{bmatrix} \left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k-1)} j_k(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_{k-1}(x,z) \\ j_{k-1}(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k-1)} j_{k-2}(x,z) \end{bmatrix}$$

$$\hat{\mathrm{J}}_{k+1}(\mathrm{x},\mathrm{z}) = \begin{cases} \begin{bmatrix} \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^1 j_{k+2}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_{k+1}(\mathrm{x},\mathrm{z}) \\ j_{k+1}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^1 j_k(\mathrm{x},\mathrm{z}) \end{bmatrix} & if\ k\ is\ even \\ \begin{bmatrix} j_{k+2}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_{k+1}(\mathrm{x},\mathrm{z}) \\ j_{k+1}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z}) j_k(\mathrm{x},\mathrm{z}) \end{bmatrix} & if\ k\ is\ odd \end{cases}$$

Hence, we end up with the conclusion that it is true for *n=k+1*. Similar result can be proved for even *k*. Hence the result.

Theorem 2.2: For any non-negative integer n, determinant can be given as

$$\det[\hat{\mathrm{J}}_n] = \left(2h(\mathrm{z})\right)^n \left(-\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(n)}$$

Proof: we know that

$$\det[\hat{\mathrm{J}}_0] = \det\begin{bmatrix} j_1(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_0(\mathrm{x},\mathrm{z}) \\ j_0(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z}) j_{-1}(\mathrm{x},\mathrm{z}) \end{bmatrix} = 1$$

$$\det[\hat{\mathrm{J}}_1] = \det\begin{bmatrix} \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(1)} j_2(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_1(\mathrm{x},\mathrm{z}) \\ j_1(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(1)} j_0(\mathrm{x},\mathrm{z}) \end{bmatrix} = \det\begin{bmatrix} v(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} \\ 1 & 0 \end{bmatrix} = -2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}$$

$$\det[\hat{\mathrm{J}}_2] = \det\begin{bmatrix} \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(2)} j_3(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_2(\mathrm{x},\mathrm{z}) \\ j_2(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(2)} j_1(\mathrm{x},\mathrm{z}) \end{bmatrix} = \left(2h(\mathrm{z})\right)^2$$

$$\det[\hat{\mathrm{J}}_3] = \det\begin{bmatrix} \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(3)} j_4(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_3(\mathrm{x},\mathrm{z}) \\ j_3(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(3)} j_2(\mathrm{x},\mathrm{z}) \end{bmatrix} = -\left(2h(\mathrm{z})\right)^3 \frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}$$

$$\det[\hat{\mathrm{J}}_4] = \det\begin{bmatrix} \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(4)} j_5(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} j_4(\mathrm{x},\mathrm{z}) \\ j_4(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(4)} j_3(\mathrm{x},\mathrm{z}) \end{bmatrix} = \left(2h(\mathrm{z})\right)^4$$

on continuing the process, we can observe that

$$\det[\hat{\mathrm{J}}_n] = \left(2h(\mathrm{z})\right)^n \left(-\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(n)}$$

Theorem 2.3*: *(Cassini's Identity) For any non-negative integer n, we have

$$\left(\frac{u(\mathrm{x},\mathrm{z})}{v(\mathrm{x},\mathrm{z})}\right)^{2\xi(n)} \hat{\mathrm{J}}_{n-1}(\mathrm{x},\mathrm{z})\hat{\mathrm{J}}_{n+1}(\mathrm{x},\mathrm{z}) - \left(\frac{u(\mathrm{x},\mathrm{z})}{v(\mathrm{x},\mathrm{z})}\right)^1 \hat{\mathrm{J}}_n{}^2(\mathrm{x},\mathrm{z}) = \left(2h(\mathrm{z})\right)^{n-1}\left(-\frac{u(\mathrm{x},\mathrm{z})}{v(\mathrm{x},\mathrm{z})}\right)^{\xi(n)}$$

Proof: Proof is obvious by theorem 2.1 and theorem 2.2.

Theorem 2.4(Generating Function). For every $m \in \mathbb{N}$, the generating function of the bivariate bi-periodic Jacobsthal Matrix polynomial sequences are denoted by $K(t)$

$$K(t) = \frac{\hat{J}_0 + \hat{J}_1 t + \left(u\hat{J}_1 - \left(uv + 2h(z)\right)\hat{J}_0\right)t^2 + +2h(z)[v\hat{J}_0 - \hat{J}_1]t^3}{1 - \left(u(x,z)v(x,z) + 4h(z)\right)t^2 + (2h(z))^2 t^4}$$

Proof: $K(t)$ is representing in the form of power series

$$K(t) = \sum_{m=0}^{\infty} \hat{J}_n(x,z)t^n$$

We can write $K(t)$ as $K(t) = K_0(t) + K_1(t)$

Where the$K_0(t)$ is even part of the series

$$K_0(t) = \hat{J}_0(x,z) + \hat{J}_2(x,z)t^2 + \cdots = \sum_{m=0}^{\infty} \hat{J}_{2n}(x,z)t^{2n}$$

And $K_1(t)$ is odd part of the series

$$K_1(t) = \hat{J}_1(x,z)t + \hat{J}_3(x,z)t^3 + \cdots = \sum_{m=0}^{\infty} \hat{J}_{2n+1}(x,z)t^{2n+1}$$

Now consider the even part of the series

$$K_0(t) = \hat{J}_0(x,z) + \hat{J}_2(x,z)t^2 + \cdots = \textstyle\sum_{n=0}^{\infty} \hat{J}_{2n}(x,z)t^{2n} \qquad (2.4.1)$$

by Lemma 2.4

$$\hat{J}_{2n}(x,z) = \left(u(x,z)v(x,z) + 4h(z)\right)\hat{J}_{2n-2}(x,z) - (2h(z))^2\hat{J}_{2n-4}(x,z)$$

on substituting the above value of $\hat{J}_{2n}(x,z)$ in (2.4.1)

$$K_0(t) = \hat{J}_0(x,z) + \hat{J}_2(x,z)t^2 + \sum_{n=2}^{\infty}\left[\left(u(x,z)v(x,z) + 4h(z)\right)j_{2n-2}(x,z) - (2h(z))^2 j_{2n-4}(x,z)\right]t^{2n}$$

On solving, we'll get

$$K_0(t) = \frac{\hat{J}_0(x,z) + \hat{J}_2(x,z)t^2 - \left(u(x,z)v(x,z) + 4h(z)\right)\hat{J}_0(x,z)t^2}{1 - \left(u(x,z)v(x,z) + 4h(z)\right)t^2 + (2h(z))^2 t^4}$$

Similarly, we can find

$$K_1(t) = \frac{\hat{J}_1(x,z)t + \hat{J}_3(x,z)t^3 - \left(u(x,z)v(x,z) + 4h(z)\right)\hat{J}_1(x,z)t^3}{1 - \left(u(x,z)v(x,z) + 4h(z)\right)t^2 + (2h(z))^2 t^4}$$

Since put the values of $\hat{J}_2$ *and* $\hat{J}_3$ in $K_0(t)$ and $K_1(t)$. After, this process, let us add odd and even series to get

$K(t) = K_0(t) + K_1(t)$, we get

$$K(t) = \frac{\hat{J}_0 + \hat{J}_1 t + (u\hat{J}_1 - \left(uv + 2h(z)\hat{J}_0\right)t^2 + +2h(z)\left[v\hat{J}_0 - \hat{J}_1\right]t^3}{1 - \left(u(x,z)v(x,z) + 4h(z)\right)t^2 + \left(2h(z)\right)^2 t^4}$$

Theorem 2.2 (Binet's formula): For every $\boldsymbol{n} \in \mathbb{N}$, the Binet's formula of the bivariate bi-periodic Jacobsthal Matrix polynomial sequences is given by

$$\hat{J}_n(x, z) = P(x, z)\left(\lambda_1{}^n - \lambda_2{}^n\right) + Q(x, z)\left(\lambda_1{}^{2\left\lfloor\frac{n}{2}\right\rfloor+2} - \lambda_2{}^{2\left\lfloor\frac{n}{2}\right\rfloor+2}\right) \quad (2.2.1)$$

$$P(x, z) = \frac{[\hat{J}_1(x,z) - \hat{J}_0(x,z)v(x,z)]^{\xi(n)}\ [u(x,z)\hat{J}_1(x,z) - 2h(z)\hat{J}_0(x,z) - u(x,z)v(x,z)\hat{J}_0(x,z)]^{1-\xi(n)}}{(u(x,z)v(x,z))^{\left\lfloor\frac{n}{2}\right\rfloor}(\lambda_1 - \lambda_2)}$$ and

$$Q(x, z) = \frac{v(x,z)^{\xi(n)}\hat{J}_0(x,z)}{\left(u(x,z)v(x,z)\right)^{\left\lfloor\frac{n}{2}\right\rfloor+1}(\lambda_1 - \lambda_2)}$$

where the parity function is $\xi(n) = n - 2\left\lfloor\frac{n}{2}\right\rfloor$.

Proof: Parity function $\xi(n)$, can be expressed as

$$\xi(n) = \begin{cases} 0 \ if\ n\ is\ even \\ 1\ \ if\ n\ is\ odd \end{cases}$$

we can find Binet's formula using two step method

1. Take generating function and using partial fraction decomposition method.

2. After using partial fraction apply Maclaurin's Series expansion.

Let us apply step 1 to get:

$$K(t) = \frac{1}{(2h(z))^2(\lambda_1 - \lambda_2)}\left(\frac{\begin{matrix}\left[2h(z)\lambda_1[v(x, z)\hat{J}_0(x, z) - \hat{J}_1(x, z)] + \left(2h(z)\right)^2[v(x, z)\hat{J}_0(x, z)]\right]t \\ + \\ +(\lambda_1 + 2h(z))\left[u(x, z)\hat{J}_1 - [u(x, z)v(x, z) + 4h(z)]\hat{J}_0(x, z)\right] \\ +\left(2h(z)\right)^2\hat{J}_0\end{matrix}}{t^2 - \frac{(\lambda_1 + 2h(z))}{(2h(z))^2}} - \frac{\begin{matrix}\left[2h(z)\lambda_2[v(x, z)\hat{J}_0(x, z) - \hat{J}_1(x, z)] + \left(2h(z)\right)^2[v(x, z)\hat{J}_0(x, z)]\right]t \\ +(\lambda_2 + 2h(z))\left[u(x, z)\hat{J}_1 - [u(x, z)v(x, z) + 4h(z)]\hat{J}_0(x, z)\right] \\ +\left(2h(z)\right)^2\hat{J}_0\end{matrix}}{t^2 - \frac{(\lambda_2 + 2h(z))}{(2h(z))^2}}\right)$$

We know that $K(t) = K_0(t) + K_1(t)$

Where $K_0(t)$ is even term series and $K_1(t)$ is odd term series and after separating series, we can use Maclaurin's Series expansion

Where $K_0(t)$ is even term series

$$\mathrm{K}_0(t)=\frac{-1}{(2h(z))^2(\lambda_1-\lambda_2)}\Bigg[\sum_{n=0}^{\infty}(\lambda_1+2h(z))\big[u(x,z)\hat{J}_1-[u(x,z)v(x,z)+4h(z)]\hat{J}_0(x,z)\big]+\big(2h(z)\big)^2\hat{J}_0\Bigg]\left(\frac{(\lambda_1+2h(z))}{(2h(z))^2}\right)^{-n-1}t^{2n}-\Bigg[\sum_{n=0}^{\infty}(\lambda_2+2h(z))\big[u(x,z)\hat{J}_1-[u(x,z)v(x,z)+4h(z)]\hat{J}_0(x,z)\big]+\big(2h(z)\big)^2\hat{J}_0\Bigg]\left(\frac{(\lambda_2+2h(z))}{(2h(z))^2}\right)^{-n-1}t^{2n}$$

$$\mathrm{K}_0(t)=-\frac{(2h(z))^{2n+2}}{(2h(z))^2(\lambda_1-\lambda_2)}\sum_{n=0}^{\infty}\Bigg[\Big[(\lambda_1+2h(z))\Big[u(x,z)\hat{J}_1-\big[u(x,z)v(x,z)+2h(z)\big)\hat{J}_0(x,z)\big]\Big]+\big(2h(z)\big)^2\hat{J}_0\Big]\frac{1}{(\lambda_1+2h(z))^{n+1}}-\Big[(\lambda_2+2h(z))\Big[u(x,z)\hat{J}_1-\big[u(x,z)v(x,z)+2h(z)\big)\hat{J}_0(x,z)\big]\Big]+\big(2h(z)\big)^2\hat{J}_0\Big]\frac{1}{(\lambda_2+2h(z))^{n+1}}\Bigg]t^{2n}$$

$$\mathrm{K}_0(t)=-\frac{(2h(z))^{2n}}{(\lambda_1-\lambda_2)}\sum_{n=0}^{\infty}\left(\frac{(\lambda_2+2h(z))^{n+1}\Big((\lambda_1+2h(z))\Big(u(x,z)\hat{J}_1-\big(u(x,z)v(x,z)+4h(z)\big)\hat{J}_0(x,z)\Big)+\big(2h(z)\big)^2\hat{J}_0\Big)-(\lambda_1+2h(z))^{n+1}\Big((\lambda_2+2h(z))\Big(u(x,z)\hat{J}_1-\big(u(x,z)v(x,z)+4h(z)\big)\hat{J}_0(x,z)\Big)+\big(2h(z)\big)^2\hat{J}_0\Big)}{(\lambda_1+2h(z))^{n+1}(\lambda_2+2h(z))^{n+1}}\right)t^{2n}$$

And using identities$(\lambda_2+2h(z))(\lambda_2+2h(z))=(2h(z))^2$, $(\lambda_2+2h(z))=\frac{(\lambda_2)^2}{u(x,z)v(x,z)}$ and $(\lambda_1+2h(z))=\frac{(\lambda_1)^2}{u(x,z)v(x,z)}$ and we get

$$\mathrm{K}_0(t)=\frac{-1}{(2h(z))^2(\lambda_1-\lambda_2)}\sum_{n=0}^{\infty}\left(\begin{array}{c}\frac{(\lambda_2)^{2n}(2h(z))^2}{\big(u(x,z)v(x,z)\big)^n}\Big(\Big(u(x,z)\hat{J}_1-\big(u(x,z)v(x,z)+4h(z)\big)\hat{J}_0(x,z)\Big)\Big)\\+\frac{(\lambda_2)^{2n+2}(2h(z))^2}{\big(u(x,z)v(x,z)\big)^{n+1}}\hat{J}_0\\-\frac{(\lambda_1)^{2n}(2h(z))^2}{\big(u(x,z)v(x,z)\big)^n}\Big(\Big(u(x,z)\hat{J}_1-\big(u(x,z)v(x,z)+4h(z)\big)\hat{J}_0(x,z)\Big)\Big)\\-\frac{(\lambda_1)^{2n+2}(2h(z))^2}{\big(u(x,z)v(x,z)\big)^{n+1}}\hat{J}_0\end{array}\right)t^{2n}$$

$$\mathrm{K}_0(t)=\frac{-1}{(\lambda_1-\lambda_2)}\sum_{n=0}^{\infty}\left(\begin{array}{c}\Big(\Big(u(x,z)\hat{J}_1-\big(u(x,z)v(x,z)+4h(z)\big)\hat{J}_0(x,z)\Big)\Big)\left(\frac{(\lambda_2)^{2n}}{\big(u(x,z)v(x,z)\big)^n}-\frac{(\lambda_1)^{2n}}{\big(u(x,z)v(x,z)\big)^n}\right)\\+\hat{J}_0\left(\frac{(\lambda_2)^{2n+2}}{\big(u(x,z)v(x,z)\big)^{n+1}}-\frac{(\lambda_1)^{2n+2}}{\big(u(x,z)v(x,z)\big)^{n+1}}\right)\end{array}\right)t^{2n}$$

$$K_0(t) = \sum_{n=0}^{\infty}\left(\frac{1}{\left(u(x,z)v(x,z)\right)^n}\left(\left(u(x,z)\hat{J}_1 - \left(u(x,z)v(x,z) + 4h(z)\right)\hat{J}_0(x,z)\right)\right)\left(\frac{{\lambda_1}^{2n} - {\lambda_2}^{2n}}{\lambda_1 - \lambda_2}\right) + \hat{J}_0\frac{1}{\left(u(x,z)v(x,z)\right)^{n+1}}\left(\frac{{\lambda_1}^{2n+2} - {\lambda_2}^{2n+2}}{\lambda_1 - \lambda_2}\right)\right)t^{2n}$$

Similarly, we can find odd series

$$K_1(t)=\frac{-1}{(2h(z))^2(\lambda_1-\lambda_2)}\left[\sum_{n=0}^{\infty}\{2h(z)\lambda_1[v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)] + \left(2h(z)\right)^2[v(x,z)\hat{J}_0(x,z)]\}\right]\left(\frac{(\lambda_1+2h(z))}{(2h(z))^2}\right)^{-n-1}t^{2n+1} - \left[\{2h(z)\lambda_2[v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)] + \left(2h(z)\right)^2[v(x,z)\hat{J}_0(x,z)]\}\right]\left(\frac{(\lambda_2+2h(z))}{(2h(z))^2}\right)^{-n-1}t^{2n+1}$$

$$K_1(t) = -\frac{(2h(z))^{2n+2}}{(2h(z))^2(\lambda_1 - \lambda_2)}\sum_{n=0}^{\infty}\left[\left(2h(z)\lambda_1[v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)] + \left(2h(z)\right)^2[v(x,z)\hat{J}_0(x,z)]\right)\frac{1}{(\lambda_1 + 2h(z))^{n+1}} - \left(2h(z)\lambda_2[v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)] + \left(2h(z)\right)^2[v(x,z)\hat{J}_0(x,z)]\right)\frac{1}{(\lambda_2 + 2h(z))^{n+1}}\right]t^{2n+1}$$

$$K_1(t) = -\frac{(2h(z))^{2n}}{(\lambda_1 - \lambda_2)}\sum_{n=0}^{\infty}\left(\frac{(\lambda_2 + 2h(z))^{n+1}\left(2h(z)\lambda_1[v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)] + \left(2h(z)\right)^2[v(x,z)\hat{J}_0(x,z)]\right) - (\lambda_1 + 2h(z))^{n+1}\left(2h(z)\lambda_2[v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)] + \left(2h(z)\right)^2[v(x,z)\hat{J}_0(x,z)]\right)}{(\lambda_1 + 2h(z))^{n+1}(\lambda_2 + 2h(z))^{n+1}}\right)t^{2n+1}$$

And using identities$(\lambda_1)(\lambda_2 + 2h(z)) = -2h(z)(\lambda_2)$,$(\lambda_2)(\lambda_1 + 2h(z)) = -2h(z)(\lambda_1)$, $(\lambda_2 + 2h(z)) = \frac{(\lambda_2)^2}{u(x,z)v(x,z)}$ and $(\lambda_1 + 2h(z)) = \frac{(\lambda_1)^2}{u(x,z)v(x,z)}$ and we get

$$K_1(t)=\frac{1}{(2h(z))^2(\lambda_1-\lambda_2)}\sum_{n=0}^{\infty}\left(\frac{(\lambda_2)^{2n+1}(2h(z))^2}{\left(u(x,z)v(x,z)\right)^n}\left([v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)]\right) - \frac{(\lambda_2)^{2n+2}(2h(z))^2}{\left(u(x,z)v(x,z)\right)^{n+1}}[v(x,z)\hat{J}_0(x,z)] - \frac{(\lambda_1)^{2n+1}(2h(z))^2}{\left(u(x,z)v(x,z)\right)^n}\left([v(x,z)\hat{J}_0(x,z) - \hat{J}_1(x,z)]\right) + \frac{(\lambda_1)^{2n+2}(2h(z))^2}{\left(u(x,z)v(x,z)\right)^{n+1}}[v(x,z)\hat{J}_0(x,z)]\right)t^{2n+1}$$

$$K_1(t) = \sum_{n=0}^{\infty}\left[\frac{1}{\left(u(x,z)v(x,z)\right)^n}[\hat{J}_1(x,z) - [v(x,z)\hat{J}_0(x,z)]]\left[\frac{{\lambda_1}^{2n+1} - {\lambda_2}^{2n+1}}{\lambda_1 - \lambda_2}\right] + \hat{J}_0\frac{v(x,z)}{\left(u(x,z)v(x,z)\right)^{n+1}}\left[\frac{{\lambda_1}^{2n+2} - {\lambda_2}^{2n+2}}{\lambda_1 - \lambda_2}\right]\right]t^{2n+1}$$

So, we have Binet's formula

$$\hat{J}_n(x,z) = P(x,z)\,({\lambda_1}^n - {\lambda_2}^n) + Q(x,z)({\lambda_1}^{2\left\lfloor\frac{n}{2}\right\rfloor+2} - {\lambda_2}^{2\left\lfloor\frac{n}{2}\right\rfloor+2})$$

$P(x,z)= \frac{[\hat{J}_1(x,z)-\hat{J}_0(x,z)v(x,z)]^{\xi(n)}\ [u(x,z)\hat{J}_1(x,z)-2h(z)\hat{J}_0(x,z)-v(x,z)\hat{J}_0(x,z)]^{1-\xi(n)}}{(u(x,z)v(x,z))^{\left\lfloor\frac{n}{2}\right\rfloor}(\lambda_1-\lambda_2)}$ and $Q(x,z) = \frac{v(x,z)^{\xi(n)}\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\left\lfloor\frac{n}{2}\right\rfloor+1}(\lambda_1-\lambda_2)}$

Theorem 2.3 (SUMMATION FORMULA) For any positive integer n, such that

$$\sum_{i=0}^{n-1}\hat{J}_i = \frac{\hat{J}_n\left[2h(z)-1-u(x,z)^{\xi(n)}v(x,z)^{1-\xi(n)}\right]+2h(z)\,\hat{J}_{n-1}\left[2h(z)-u(x,z)^{1-\xi(n)}v(x,z)^{\xi(n)}-1\right]+\hat{J}_1[1+2h(z))+u(x,z)] +\hat{J}_0[1-2h(z)-u(x,z)v(x,z)-2h(z)v(x,z)]}{[(2h(z)-1)^2-u(x,z)v(x,z)]}$$

Proof: we know that

$$\sum_{i=0}^{n-1}\hat{J}_i=\sum_{i=0}^{\frac{n-2}{2}}\hat{J}_{2i}+\sum_{i=0}^{\frac{n-2}{2}}\hat{J}_{2i+1}$$

$$= \sum_{i=0}^{\frac{n-2}{2}}\left[\frac{1}{(u(x,z)v(x,z))^i}\left[\left[u(x,z)\hat{J}_1-[u(x,z)v(x,z)+4h(z)]\hat{J}_0(x,z)\right]\right]\left[\frac{{\lambda_1}^{2i}-{\lambda_2}^{2i}}{\lambda_1-\lambda_2}\right]+\hat{J}_0\frac{1}{(u(x,z)v(x,z))^{i+1}}\left[\frac{{\lambda_1}^{2i+2}-{\lambda_2}^{2i+2}}{\lambda_1-\lambda_2}\right]\right]+\sum_{i=0}^{\frac{n-2}{2}}\left[\frac{1}{(u(x,z)v(x,z))^i}\left[\hat{J}_1(x,z)-v(x,z)\hat{J}_0(x,z)\right]\left[\frac{{\lambda_1}^{2i+1}-{\lambda_2}^{2i+1}}{\lambda_1-\lambda_2}\right]+\hat{J}_0\frac{v(x,z)}{(u(x,z)v(x,z))^{i+1}}\left[\frac{{\lambda_1}^{2i+2}-{\lambda_2}^{2i+2}}{\lambda_1-\lambda_2}\right]\right]$$

If n is even, we can simplify with the help of arithmetic mean

$$\sum_{i=0}^{n-1}\hat{J}_i=\frac{\left(u(x,z)\hat{J}_1-(u(x,z)v(x,z)+4h(z))\hat{J}_0(x,z)\right)}{(u(x,z)v(x,z))^{\frac{n}{2}-1}(\lambda_1-\lambda_2)}\left[\frac{{\lambda_1}^n-(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_1}^2-(u(x,z)v(x,z))}-\frac{{\lambda_2}^n-(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_2}^2-(u(x,z)v(x,z))}\right]$$

$$+\frac{\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1-\lambda_2)}\left[\frac{{\lambda_1}^{n+2}-{\lambda_1}^2(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_1}^2-(u(x,z)v(x,z))}-\frac{{\lambda_2}^{n+2}-{\lambda_2}^2(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_2}^2-(u(x,z)v(x,z))}\right]$$

$$+\frac{\hat{J}_1(x,z)-v(x,z)\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}-1}(\lambda_1-\lambda_2)}\left[\frac{{\lambda_1}^{n+1}-\lambda_1(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_1}^2-(u(x,z)v(x,z))}-\frac{{\lambda_2}^{n+1}-\lambda_2(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_2}^2-(u(x,z)v(x,z))}\right]$$

$$+\frac{v(x,z)\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1-\lambda_2)}\left[\frac{{\lambda_1}^{n+2}-{\lambda_1}^2(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_1}^2-(u(x,z)v(x,z))}-\frac{{\lambda_2}^{n+2}-{\lambda_2}^2(u(x,z)v(x,z))^{\frac{n}{2}}}{{\lambda_2}^2-(u(x,z)v(x,z))}\right]$$

$$= \frac{\left(u(x,z)\hat{J}_1-(u(x,z)v(x,z)+4h(z))\hat{J}_0(x,z)\right)}{(u(x,z)v(x,z))^{\frac{n}{2}-1}(\lambda_1-\lambda_2)}\left[\frac{\left({\lambda_1}^n-(u(x,z)v(x,z))^{\frac{n}{2}}\right)({\lambda_2}^2-(u(x,z)v(x,z)))-\left({\lambda_2}^n-(u(x,z)v(x,z))^{\frac{n}{2}}\right)({\lambda_1}^2-(u(x,z)v(x,z)))}{({\lambda_1}^2-(u(x,z)v(x,z)))({\lambda_2}^2-(u(x,z)v(x,z)))}\right]$$

$$+\frac{\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1-\lambda_2)}\left[\frac{\left({\lambda_1}^{n+2}-{\lambda_1}^2(u(x,z)v(x,z))^{\frac{n}{2}}\right)({\lambda_2}^2-(u(x,z)v(x,z)))-\left({\lambda_2}^{n+2}-{\lambda_2}^2(u(x,z)v(x,z))^{\frac{n}{2}}\right)({\lambda_1}^2-(u(x,z)v(x,z)))}{({\lambda_1}^2-(u(x,z)v(x,z)))({\lambda_2}^2-(u(x,z)v(x,z)))}\right]$$

$$+\frac{\hat{J}_1(x,z)-v(x,z)\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}-1}(\lambda_1-\lambda_2)}\left[\frac{\left(\lambda_1^{n+1}-\lambda_1(u(x,z)v(x,z))^{\frac{n}{2}}\right)(\lambda_2^2-(u(x,z)v(x,z)))-\left(\lambda_2^{n+1}-\lambda_2(u(x,z)v(x,z))^{\frac{n}{2}}\right)(\lambda_1^2-(u(x,z)v(x,z)))}{(\lambda_1^2-(u(x,z)v(x,z)))(\lambda_2^2-(u(x,z)v(x,z)))}\right]$$

$$+\frac{v(x,z)\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1-\lambda_2)}\left[\frac{\left(\lambda_1^{n+2}-\lambda_1^2(u(x,z)v(x,z))^{\frac{n}{2}}\right)(\lambda_2^2-(u(x,z)v(x,z)))-\left(\lambda_2^{n+2}-\lambda_2^2(u(x,z)v(x,z))^{\frac{n}{2}}\right)(\lambda_1^2-(u(x,z)v(x,z)))}{(\lambda_1^2-(u(x,z)v(x,z)))(\lambda_2^2-(u(x,z)v(x,z)))}\right]$$

$$=\frac{\left(u(x,z)\hat{J}_1-(u(x,z)v(x,z)+4h(z))\hat{J}_0(x,z)\right)}{(u(x,z)v(x,z))^{\frac{n}{2}-1}(\lambda_1-\lambda_2)}\left[\frac{\lambda_1^2\lambda_2^2(\lambda_1^{n-2}-\lambda_2^{n-2})+(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1^2-\lambda_2^2)-(u(x,z)v(x,z))(\lambda_1^n-\lambda_2^n)}{(\lambda_1^2-(u(x,z)v(x,z)))(\lambda_2^2-(u(x,z)v(x,z)))}\right]$$

$$+\frac{\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1-\lambda_2)}\left[\frac{\lambda_1^2\lambda_2^2(\lambda_1^n-\lambda_2^n)+(u(x,z)v(x,z))^{\frac{n}{2}+1}(\lambda_1^2-\lambda_2^2)-(u(x,z)v(x,z))(\lambda_1^{n+2}-\lambda_2^{n+2})}{(\lambda_1^2-(u(x,z)v(x,z)))(\lambda_2^2-(u(x,z)v(x,z)))}\right]$$

$$+$$

$$\frac{\hat{J}_1(x,z)-v(x,z)\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}-1}(\lambda_1-\lambda_2)}\left[\frac{\lambda_1^2\lambda_2^2(\lambda_1^{n-1}-\lambda_2^{n-1})+(u(x,z)v(x,z))^{\frac{n}{2}+1}(\lambda_1^1-\lambda_2^1)-(u(x,z)v(x,z))(\lambda_1^{n+1}-\lambda_2^{n+1})-(\lambda_1\lambda_2)(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1^1-\lambda_2^1)}{(\lambda_1^2-(u(x,z)v(x,z)))(\lambda_2^2-(u(x,z)v(x,z)))}\right]$$

$$+\frac{v(x,z)\hat{J}_0(x,z)}{(u(x,z)v(x,z))^{\frac{n}{2}}(\lambda_1-\lambda_2)}\left[\frac{\lambda_1^2\lambda_2^2(\lambda_1^n-\lambda_2^n)+(u(x,z)v(x,z))^{\frac{n}{2}+1}(\lambda_1^2-\lambda_2^2)-(u(x,z)v(x,z))(\lambda_1^{n+2}-\lambda_2^{n+2})}{(\lambda_1^2-(u(x,z)v(x,z)))(\lambda_2^2-(u(x,z)v(x,z)))}\right]$$

Use Binet's formula and $\left(\lambda_1^2-(u(x,z)v(x,z))\right)\left(\lambda_2^2-(u(x,z)v(x,z))\right)=((2h(z)-1)^2-(u(x,z)v(x,z)))$we get

$$=\frac{(2h(z))^2\hat{J}_{n-2}+(2h(z))^2\hat{J}_{n-1}-\hat{J}_n-\hat{J}_{n+1}+\hat{J}_1(1+2h(z))+u(x,z))+\hat{J}_0(1-2h(z)-u(x,z)v(x,z)-2h(z)v(x,z))}{(2h(z)-1)^2-(u(x,z)v(x,z)))}$$

Here n is even so we use

$\hat{J}_n = u(x,z)\hat{J}_{n-1}+2h(z)\hat{J}_{n-2}$ and $\hat{J}_{n+1} = v(x,z)\hat{J}_n+2h(z)\hat{J}_{n-1}$ to get

$$\frac{2h(z)\hat{J}_{n-1}(2h(z)-u(x,z)-1)+\hat{J}_n(2h(z)-v(x,z)-1)+\hat{J}_1\left(1+2h(z))+u(x,z)\right)+\hat{J}_0\left(1-2h(z)-u(x,z)v(x,z)-2h(z)v(x,z)\right)}{(2h(z)-1)^2-(u(x,z)v(x,z)))}$$

Similarly, we can solve when n is odd

$$\frac{2h(z)\hat{J}_{n-1}(2h(z)-v(x,z)-1)+\hat{J}_n(2h(z)-u(x,z)-1)+\hat{J}_1\left(1+2h(z))+u(x,z)\right)+\hat{J}_0\left(1-2h(z)-u(x,z)v(x,z)-2h(z)v(x,z)\right)}{(2h(z)-1)^2-(u(x,z)v(x,z)))}$$

Hence, we get summation formula

$$\sum_{i=0}^{n-1}\hat{J}_i = \frac{\hat{J}_n\left[2h(z)-1-u(x,z)^{\xi(n)}v(x,z)^{1-\xi(n)}\right]+2h(z)\,\hat{J}_{n-1}\left[2h(z)-u(x,z)^{1-\xi(n)}v(x,z)^{\xi(n)}-1\right]+\hat{J}_1[1+2h(z))+u(x,z)] + \hat{J}_0[1-2h(z)-u(x,z)v(x,z)-2h(z)v(x,z)]}{[(2h(z)-1)^2-u(x,z)v(x,z)]}$$

Theorem 2.4 (Generating Function and Binet's Formula). For every $n \in \mathbb{N}$, $\boldsymbol{n^{th}}$ term of $\{\boldsymbol{j_n}(x,z)\}_{\boldsymbol{n=0}}^{\infty}$,the generating function and binet's formula of the bivariate bi-periodic Jacobsthal are given by

$$J(t)=\frac{t(1+u(x,z)t-2h(z)t^2)}{1-\left(u(x,z)v(x,z)+4h(z)\right)t^2+(2h(z))^2t^4}$$

$$j_n(x,z)=\frac{u(x,z)^{1-\xi(n)}}{(u(x,z)v(x,z))^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{{\lambda_1}^n-{\lambda_2}^n}{\lambda_1-\lambda_2}\right)$$

Theorem2.5: let $\hat{H}_q$ generating matrix be as in equation (1.1.2). Then, we have

$$(\hat{H}_q)^n=\left(\frac{v(x,z)}{u(x,z)}\right)^{\lfloor\frac{n}{2}\rfloor}\begin{bmatrix}\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(n)}j_{n+1}(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_n(x,z)\\ j_n(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(n)}j_{n-1}(x,z)\end{bmatrix} \qquad (2.5.1)$$

Where $\xi(n)=n-2\left\lfloor\frac{n}{2}\right\rfloor$ and $j_n(x,z)$ is nth Bi-variate Bi-periodic Jacobsthal polynomials.

Proof: We use Principle Mathematical Induction on n, we can write

$$\hat{H}_q(x,z)=\begin{bmatrix}v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}\\ 1 & 0\end{bmatrix}=\left(\frac{v(x,z)}{u(x,z)}\right)^{\lfloor\frac{1}{2}\rfloor}\begin{bmatrix}\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(1)}j_2(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_1(x,z)\\ j_1(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(1)}j_0(x,z)\end{bmatrix}$$

$$\hat{H}_q(x,z)=\begin{bmatrix}\left(\frac{v(x,z)}{u(x,z)}\right)j_2(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_1(x,z)\\ j_1(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_0(x,z)\end{bmatrix}$$

where $j_0(x,z)=0$, $j_1(x,z)=1$ and $j_2(x,z)=u(x,z)$

$$\hat{H}_q(x,z)=\begin{bmatrix}v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}\\ 1 & 0\end{bmatrix}$$

$$(\hat{H}_q)^2=$$

$$\begin{bmatrix}(v(x,z))^2+2h(z)\frac{v(x,z)}{u(x,z)} & 2h(z)\frac{(v(x,z))^2}{u(x,z)}\\ v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}\end{bmatrix}=\left(\frac{v(x,z)}{u(x,z)}\right)^{\lfloor\frac{2}{2}\rfloor}\begin{bmatrix}\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(2)}j_3(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_2(x,z)\\ j_2(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(2)}j_1(x,z)\end{bmatrix}$$

$$=\frac{v(x,z)}{u(x,z)}\begin{bmatrix}j_3(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_2(x,z)\\ j_2(x,z) & 2h(z)j_1(x,z)\end{bmatrix}$$

where $j_0(x,z)=0$, $j_1(x,z)=1$, $j_2(x,z)=u(x,z)$ and $j_3(x,z)=u(x,z)\,v(x,z)+2h(z)$

$$=\begin{bmatrix}(v(x,z))^2+2h(z)\frac{v(x,z)}{u(x,z)} & 2h(z)\frac{(v(x,z))^2}{u(x,z)}\\ v(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}\end{bmatrix}$$

Which show that equation (2.5.1) are true for $n=1$ and $n=2$. Now we assume that it is true for $n=k$ that is

$$(\hat{H}_q)^k=\left(\frac{v(x,z)}{u(x,z)}\right)^{\lfloor\frac{k}{2}\rfloor}\begin{bmatrix}\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k)}j_{k+1}(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)}j_k(x,z)\\ j_k(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(k)}j_{k-1}(x,z)\end{bmatrix}$$

If we assume that k is even, by using properties of the Bi-variate Bi-periodic Jacobsthal polynomials, we get

$$\left(\hat{\mathrm{H}}_q\right)^{k+2} = \left(\hat{\mathrm{H}}_q\right)^{k}\left(\hat{\mathrm{H}}_q\right)^{2} = \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\frac{k+2}{2}}\begin{bmatrix} j_{k+3}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}j_{k+2}(\mathrm{x},\mathrm{z}) \\ j_{k+2}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})j_{k+1}(\mathrm{x},\mathrm{z}) \end{bmatrix}$$

Similarly, k is odd

$$\left(\hat{\mathrm{H}}_q\right)^{k+2} = \left(\hat{\mathrm{H}}_q\right)^{k}\left(\hat{\mathrm{H}}_q\right)^{2} = \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\frac{k+1}{2}}\begin{bmatrix} \frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}j_{k+3}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}j_{k+2}(\mathrm{x},\mathrm{z}) \\ j_{k+2}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}j_{k+1}(\mathrm{x},\mathrm{z}) \end{bmatrix}$$

If we put together, we get

$$\left(\hat{\mathrm{H}}_q\right)^{k+2} = \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\left\lfloor\frac{k+2}{2}\right\rfloor}\begin{bmatrix} \left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(k+2)} j_{k+3}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}j_{k+2}(\mathrm{x},\mathrm{z}) \\ j_{k+2}(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\left(\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right)^{\xi(k+2)} j_{k+1}(\mathrm{x},\mathrm{z}) \end{bmatrix}$$

Theorem 2.6: For n be any integer. The Binet's formula of Bi-variate Bi-periodic Jacobsthal polynomials is

$$j_n(\mathrm{x},\mathrm{z}) = \frac{u(\mathrm{x},\mathrm{z})^{1-\xi(n)}}{\left(u(\mathrm{x},\mathrm{z})v(\mathrm{x},\mathrm{z})\right)^{\left\lfloor\frac{n}{2}\right\rfloor}}\left(\frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}\right)$$

Characteristics equation is

$$\lambda^2 - u(\mathrm{x},\mathrm{z})v(\mathrm{x},\mathrm{z})\lambda - 2u(\mathrm{x},\mathrm{z})v(\mathrm{x},\mathrm{z})h(\mathrm{z}) = 0$$

with roots $\lambda_1(\mathrm{x},\mathrm{z})$ and $\lambda_2(\mathrm{x},\mathrm{z})$

Proof: Let the $\hat{\mathrm{H}}_q = \begin{bmatrix} v(\mathrm{x},\mathrm{z}) & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} \\ 1 & 0 \end{bmatrix}$ is generating matrix. The characteristics equation is

$$r^2 - v(\mathrm{x},\mathrm{z})r - 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} = 0$$

Then the eigen values and eigen vectors of the $\hat{\mathrm{H}}_q(\mathrm{x},\mathrm{z})$ are

$r_1 = \frac{\lambda_1(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}$ and $r_2 = \frac{\lambda_2(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}$, $p_1 = \left[2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}, -\frac{\lambda_2(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right]$ and $p_2 = \left[2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}, -\frac{\lambda_1(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})}\right]$

The generating matrix can be diagonalized by using

$$A = \mathrm{P}^{-1}\hat{\mathrm{H}}_q\mathrm{P}$$

Where

$$\mathrm{P} = \begin{bmatrix} 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} & 2h(\mathrm{z})\frac{v(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} \\ -\frac{\lambda_2(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} & -\frac{\lambda_1(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} \end{bmatrix}, A = diagonal[r_1, r_2] = \begin{bmatrix} \frac{\lambda_1(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} & 0 \\ 0 & \frac{\lambda_2(\mathrm{x},\mathrm{z})}{u(\mathrm{x},\mathrm{z})} \end{bmatrix}.$$

From properties of similar matrices, for n is any integer, we obtain

$$\left(\hat{H}_q\right)^n = \mathrm{P}A^n\mathrm{P}^{-1}$$

Thus, we get

$$(A)^n = \begin{bmatrix} \left(\frac{\lambda_1(x,z)}{u(x,z)}\right)^n & 0 \\ 0 & \left(\frac{\lambda_2(x,z)}{u(x,z)}\right)^n \end{bmatrix} \text{ and } P^{-1} = \frac{u^2}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} -\frac{\lambda_1(x,z)}{u(x,z)} & -2h(z)\frac{v(x,z)}{u(x,z)} \\ \frac{\lambda_2(x,z)}{u(x,z)} & 2h(z)\frac{v(x,z)}{u(x,z)} \end{bmatrix}$$

$$\left(\hat{H}_q\right)^n$$

$$= \frac{u^2}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} 2h(z)\frac{v(x,z)}{u(x,z)} & 2h(z)\frac{v(x,z)}{u(x,z)} \\ -\frac{\lambda_2(x,z)}{u(x,z)} & -\frac{\lambda_1(x,z)}{u(x,z)} \end{bmatrix}\begin{bmatrix} \left(\frac{\lambda_1(x,z)}{u(x,z)}\right)^n & 0 \\ 0 & \left(\frac{\lambda_2(x,z)}{u(x,z)}\right)^n \end{bmatrix}\begin{bmatrix} -\frac{\lambda_1(x,z)}{u(x,z)} & -2h(z)\frac{v(x,z)}{u(x,z)} \\ \frac{\lambda_2(x,z)}{u(x,z)} & 2h(z)\frac{v(x,z)}{u(x,z)} \end{bmatrix}$$

$$= \frac{u^2}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} 2h(z)\frac{v(x,z)}{u(x,z)} & 2h(z)\frac{v(x,z)}{u(x,z)} \\ -\frac{\lambda_2(x,z)}{u(x,z)} & -\frac{\lambda_1(x,z)}{u(x,z)} \end{bmatrix}\begin{bmatrix} -\left(\frac{\lambda_1(x,z)}{u(x,z)}\right)^{n+1} & -2h(z)\frac{v(x,z)\lambda_1{}^n}{u(x,z)^{n+1}} \\ \left(\frac{\lambda_2(x,z)}{u(x,z)}\right)^{n+1} & 2h(z)\frac{v(x,z)\lambda_2{}^n}{u(x,z)^{n+1}} \end{bmatrix}$$

$$= \frac{u^2}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} -2h(z)\frac{v(x,z)\lambda_1{}^{n+1}}{u(x,z)^{n+2}} + 2h(z)\frac{v(x,z)\lambda_1{}^{n+1}}{u(x,z)^{n+2}} & -\left(2h(z)v(x,z)\right)^2\frac{\lambda_1{}^n}{u(x,z)^{n+2}} + \left(2h(z)v(x,z)\right)^2\frac{v(x,z)\lambda_2{}^n}{u(x,z)^{n+2}} \\ \frac{\lambda_1{}^{n+1}\lambda_2(x,z)}{u(x,z)^{n+2}} - \frac{\lambda_2{}^{n+1}\lambda_1(x,z)}{u(x,z)^{n+2}} & 2h(z)\frac{v(x,z)\lambda_1{}^n\lambda_2(x,z)}{u(x,z)^{n+2}} - 2h(z)\frac{v(x,z)\lambda_2{}^n\lambda_1(x,z)}{u(x,z)^{n+2}} \end{bmatrix}$$

$$= \frac{u^2}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} 2h(z)\frac{v(x,z)(\lambda_2{}^{n+1}-\lambda_1{}^{n+1})}{(u^2)u(x,z)^n} & \left(2h(z)v(x,z)\right)^2\frac{(\lambda_2{}^n-\lambda_1{}^n)}{(u^2)u(x,z)^n} \\ \frac{\lambda_1\lambda_2(\lambda_1{}^n-\lambda_2{}^n)}{(u^2)u(x,z)^n} & 2h(z)v(x,z)\frac{\lambda_1\lambda_2(\lambda_1{}^n-\lambda_2{}^n)}{(u^2)u(x,z)^n} \end{bmatrix}$$

$$= \frac{1}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} 2h(z)\frac{v(x,z)(\lambda_2{}^{n+1}-\lambda_1{}^{n+1})}{u(x,z)^n} & \left(2h(z)v(x,z)\right)^2\frac{(\lambda_2{}^n-\lambda_1{}^n)}{u(x,z)^n} \\ \frac{\lambda_1\lambda_2(\lambda_1{}^n-\lambda_2{}^n)}{u(x,z)^n} & 2h(z)v(x,z)\frac{\lambda_1\lambda_2(\lambda_1{}^{n-1}-\lambda_2{}^{n-1})}{u(x,z)^n} \end{bmatrix}$$

$$= \frac{1}{2v(x,z)h(z)(\lambda_2-\lambda_1)}\begin{bmatrix} 2h(z)\frac{v(x,z)(\lambda_2{}^{n+1}-\lambda_1{}^{n+1})}{u(x,z)^n} & \left(2h(z)v(x,z)\right)^2\frac{(\lambda_2{}^n-\lambda_1{}^n)}{u(x,z)^n} \\ -2u(x,z)v(x,z)h(z)\frac{\lambda_1\lambda_2(\lambda_1{}^n-\lambda_2{}^n)}{u(x,z)^n} & \left(-2u(x,z)v(x,z)h(z)\right)2h(z)v(x,z)\frac{(\lambda_1{}^{n-1}-\lambda_2{}^{n-1})}{u(x,z)^n} \end{bmatrix}$$

$$= \frac{1}{(\lambda_1-\lambda_2)}\begin{bmatrix} \frac{(\lambda_1{}^{n+1}-\lambda_2{}^{n+1})}{u(x,z)^n} & \left(2h(z)\frac{v(x,z)}{u(x,z)}\right)^1\frac{(\lambda_1{}^n-\lambda_2{}^n)}{u(x,z)^{n-1}} \\ \frac{(\lambda_1{}^n-\lambda_2{}^n)}{u(x,z)^{n-1}} & 2h(z)\frac{v(x,z)}{u(x,z)}\frac{(\lambda_1{}^{n-1}-\lambda_2{}^{n-1})}{u(x,z)^{n-2}} \end{bmatrix}$$

Use Binet's formula, we get

$$\left(\hat{H}_q\right)^n = \left(\frac{v(x,z)}{u(x,z)}\right)^{\left\lfloor\frac{n}{2}\right\rfloor} \begin{bmatrix} \left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(n)} j_{n+1}(x,z) & 2h(z)\frac{v(x,z)}{u(x,z)} j_n(x,z) \\ j_n(x,z) & 2h(z)\left(\frac{v(x,z)}{u(x,z)}\right)^{\xi(n)} j_{n-1}(x,z) \end{bmatrix}$$

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