Prime Distance Labeling Of The Non-Commuting Graph Of Some Non-Abelian Groups

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***Abstract:* A graph $G(V, E)$ is said to admit a prime distance labeling if there exists an one-to-one function $h : V(G) \to Z$ such that $|h(u) - h(v)|$ is a prime number for every pair of adjacent vertices u and v in G. In this paper, we investigate if the prime distance labeling of the non-commuting graph of non-abelian groups such as symmetric group (S_n, o), dihedral group $D_{2n}, n \in N$ exists or not.**

***Keywords:* Prime Distance Labeling, Centre of Group, Non-Commuting Graph, Non-Abelian Group, Symmetric Group, Dihedral Group**

1. Introduction

All groups discussed in this paper are finite. All graphs we encounter in this paper are simple, finite, connected, non-trivial, and undirected. In 1975, Paul Erdos considered the non-commuting graph for the very first time. Later in 2014, Abdollahi et al. [1] studied certain properties of the non-commuting graph of a group. For more results, one can see [1, 2]. We use standard notations as Z for the set of all integers, $Z(G)$ for the centre of group $(G,*)$, and $\Gamma(G)$ for the non-commuting graph of group $(G,*)$. Let V and E be the vertex set and edge set of $\Gamma(G)$, respectively.

In 2013, Laison et al. [3] have defined a graph G to be a prime distance graph if there exists a one to-one labeling of its vertices given by $h : V(G) \to Z$ such that for any two adjacent vertices u and v, the integer $|h(u) - h(v)|$ is a prime and h is called a prime distance labeling of G. They also defined that $h(uv) = |h(u) - h(v)|$ and called h a prime distance labeling of G. Therefore, G is a prime distance graph if and only if there exists a prime distance labeling of G. Note that in a prime distance labeling, the vertex labels of G must be distinct, but the edge labels need not be. Also note that by this definition, $h(uv)$ may still be prime if uv is not an edge of G. For a detailed study on prime distance labeling of graphs one can refer to [3-11].

Definition 1.
A group $(G, *)$ is said to be non-abelian group if there exists at least two elements a, b in G such that $a * b \neq b * a$.
Definition 2.

The centre of a group $(G, *)$ is defined as the subset of G containing all those elements which commute with each other elements of group G under binary operation $*$. It is denoted by $Z(G) = \{x \in G: x * g = g * x, \forall g \in G\}$.

Definition 3.

A non-commuting graph of any group $(G, *)$ is denoted by $\Gamma(G)$ having the vertex set $V = G - Z(G)$ and edge set is defined by $E = \{e : e$ is edge between vertex pair (u, v) such that $u * v \neq v * u$, for $u, v \in V\}$.

Definition 4.

The set of all permutations on finite set $X = \{1, 2, 3, \dots, n\}, n \in N$ forms a group with respect to operation 'o' of composition of functions, known as symmetric group and denoted by (S_n, o), for $n \in N$.

Definition 5.

The Dihedral group $D_{2n}, n \in N$, is the group of symmetries of polygon of n −sides and is defined as $D_{2n} = \{< x, y > : x^n = e = y^2, xy = yx^{-1}\}$, where e is the identity element of group.

2. Main Results

In this section, we recall that the prime distance labeling of non-commuting graph $\Gamma(G)$ does not exist for non-abelian symmetric groups $(S_n, o), n \geq 3$.

Lemma 1. [9]

No triangular graph can possess prime distance labeling with all vertices labeled either odd integers or even integers.

2.1. Prime Distance Labeling of Non-Commuting Graph of Non-abelian Symmetric Groups

Theorem 1. [9]

The non-commuting graph $\Gamma(S_3)$ of non-abelian symmetric group (S_3, o) does not permit a prime distance labeling.

Theorem 2. [9]

The non-commuting graph $\Gamma(S_3)$, $n \geq 4$, of non-abelian symmetric group (S_n, o), $n \geq 4$, does not permit a prime distance labeling.

Corollary 1.

A prime distance labeling of non-commuting graph of Dihedral group D_6 of order 6 does not exist.

Proof.

We know that Dihedral group D_6 can be represented by: $D_6 = \{ < x, y >: x^3 = e = y^2, xy = yx^{-1} \}$, where e is the identity element of the group and D_6 is isomorphic to symmetric group (S_3, o). So, the non-commuting graph of D_6 is same as that of S_3. Hence, Theorem 1 implies that the non-commuting graph of D_6 does not admit a prime distance labeling.

Theorem 3.

The prime distance labeling of the non-commuting graph of Dihedral group of order 8 does not exist.

Proof.

The Dihedral group of order 8, D_8 is also known as Octic group which is represented by: $D_8 = \{ < x, y >: x^4 = e = y^2, xy = yx^{-1}\}$, where e is the identity elemet of group $D_8 = \{e, x, x^2, x^3, y, xy, x^2y, x^3y \}$ and the centre of $D_8, Z(D_8) = \{e, x^2\}$ which imply that e and x^2 commute with each of the remaining elements of group D_8. Firstly, we construct the non-commuting graph of D_8. Now, the vertex set V of graph $\Gamma(D_8)$ is given by $= D_8 - Z(D_8) = \{x, x^3, y, xy , x^2y, x^3y\}$. Taking $v_1 = x, v_2 = x^3, v_3 = y, v_4 = xy, v_5 = x^2y, v_6 =$

x^3y, we get $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. To find the edge set E of $\Gamma(D_8)$, one needs to find the pair of elements of V which don't commute with each other. Since, $ea = ae, \forall\, a \in D_8$, $xy = yx^{-1} \neq yx \Rightarrow xy \neq yx, x^3y = yx^{-3} = yx^{-1} = xy \Rightarrow x^3y \neq yx^3$, $x(xy) = x^2y$ & $(xy)x = yx^{-1}x = y \neq x^2y \Rightarrow x(xy) \neq (xy)x$, $x^3(xy) = x^4y = ey = y$ & $(xy)x^3 = yx^{-1}x^3 = yx^2 = x^2y \neq y \Rightarrow x^3(xy) \neq (xy)x^3$, $x(x^2y) = x^3y$ & $(x^2y)x = yx^{-2}x = yx^{-1} = xy \neq x^3y \Rightarrow x(x^2y) \neq (x^2y)x, x^3(x^2y) = x^5y = exy = xy$ & $(x^2y)x^3 = yx^{-2}x^3 = yx = x^{-1}y \neq xy \Rightarrow x^3(x^2y) \neq (x^2y)x^3$, $x(x^3y) = x^4y = ey = y$ & $(x^3y)x = yx^{-3}x = yx^{-2} = x^2y \neq y \Rightarrow x(x^3y) \neq (x^3y)x$, $x^3(x^3y) = x^6y = ex^2y = x^2y$ & $(x^3y)x^3 = yx^{-3}x^3 = ye = y \neq x^2y \Rightarrow x^3(x^3y) \neq (x^3y)x^3$. Hence, the following are the pair of adjacent vertices: $(x, y), (x, xy), (x, x^2y), (x, x^3y), (x^3, y), (x^3, xy), (x^3, x^2y), (x^3, x^3y), (xy, y), (x^3y, y), (x^2y, xy)$, and (x^3y, x^2y). Hence, the non-commuting graph of D_8 is given by:

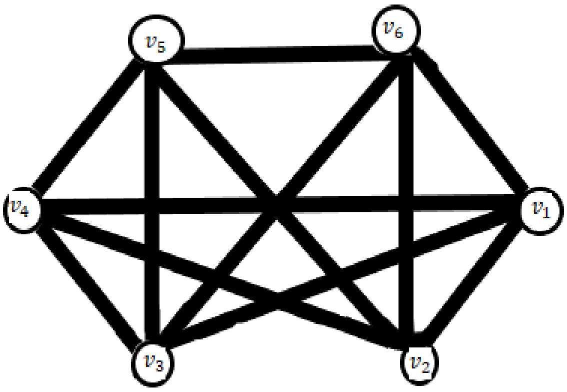

Figure 1. The Non-Commuting Graph $\Gamma(D_8)$

Let's suppose that there exists a function $\varphi: V(\Gamma(D_8)) \to \mathbb{Z}$ which determines a prime distance labeling of $\Gamma(D_8)$, i.e. $|\varphi(u) - \varphi(v)|$ is a prime number for each pair of adjacent vertices (u, v) in $\Gamma(D_8)$. Applying Lemma 1 on $\Delta v_1v_2v_6$, if we assume $\varphi(v_1), \varphi(v_2)$ to be even integers, then $\varphi(v_6)$ must be an odd integer. Say, $\varphi(v_1) = 2m$, $\varphi(v_6) = 2n + 1$, for $n, m \in \mathbb{Z}$ ..... (I)

Then, $\varphi(v_2) = 2m + 2$..... (II)

Also, $\Delta v_1v_2v_4$ implies that $\varphi(v_4)$ must be an odd integer, say $\varphi(v_4) = 2r + 1, r \in \mathbb{Z}$. Now in $\Delta v_5v_2v_4$, $\varphi(v_4)$ is an odd integer and $\varphi(v_2)$ is an even integer. So by Lemma 1, $\varphi(v_5)$ can be either an even integer or an odd integer. Therefore the following two cases arise:

Case 1: When $\varphi(v_5)$ is an even integer

Since (v_5, v_2) is a pair of adjacent vertices and $\varphi(v_2) = 2m + 2$, so, $\varphi(v_5) = 2m + 4$..... (III)

Now as v_3 is the common vertex of $\Delta v_3v_4v_5, \Delta v_1v_3v_6$, and $\Delta v_5v_3v_6$, it is obvious that $\varphi(v_3)$ can be either be an odd or even integer. If $\varphi(v_3)$ is an even integer and from Fig. 1, it is clear that the pairs of vertices (v_3, v_5) & (v_3, v_1) are connected by an edge. Since, $\varphi(v_5) = 2m + 4$ and $\varphi(v_1) = 2m$, then $\varphi(v_3)$ must be selected in such a way that both terms $|\varphi(v_3) - \varphi(v_5)|$ and $|\varphi(v_3) - \varphi(v_1)|$ must be prime number, in particularly, the even prime number 2. So, if for $(v_5) = 2m + 4, \varphi(v_3)$ should be consecutive even integer $i.e. \varphi(v_3) = 2m + 2$ or $2m + 6$, but euqation (II) implies that $\varphi(v_3) = 2m + 2$ and the prime distance labeling of vertices with integers is unique. So $\varphi(v_3) = 2m + 6$, which implies $|\varphi(v_3) - \varphi(v_1)| = 6$, which is not prime number. So, $\varphi(v_3)$ cannot be an even integer and hence, it must be an odd integer. Also Fig. 1 clears that (v_3, v_4) and (v_3, v_6) are pair of adjacent vertices. Combining the fact that $|\varphi(v_3) - \varphi(v_6)|$ & $|\varphi(v_3) - \varphi(v_4)|$ are prime numbers, particularly even prime numbers with assumption tha t $\varphi(v_3)$,

$\varphi(v_6)$ & $\varphi(v_4)$ all are odd numbers. So $(\varphi(v_3), \varphi(v_6))$ and $(\varphi(v_3), \varphi(v_4))$ must be pair of consecutive odd numbers and since the difference of any two consecutive odd integers is 2. So, we can re-write their values as $\varphi(v_6) = 2n + 1, \varphi(v_3) = 2n + 3$ and $\varphi(v_4) = 2n + 5$. Hence, we have the following labeling in this case: $\varphi(v_1) = 2m, \varphi(v_2) = 2m + 2, \varphi(v_3) = 2n + 3, \varphi(v_4) = 2n + 5, \varphi(v_5) = 2m + 4$ and $\varphi(v_6) = 2n + 1$. Since the function $\varphi: V(\Gamma(D_8)) \to \mathbb{Z}$ determines a prime distance labeling on graph shown in Fig. 1, so we have the following interpretations:

(vi,vj) →	(v_1,v_3)	(v_1,v_4)	(v_1,v_6)	(v_2,v_4)	(v_2,v_6)	(v_5,v_3)
| φ (vi)- φ (vj)|	P_1= |2(m-n)-3|	P_2= |2(m-n)-5|	P_3= |2(m-n)-1|	P_1= |2(m-n)-3|	P_4= |2(m-n)+1|	P_4= |2(m-n)+1|
(vi,vj) →	(v_5,v_4)	(v_5,v_6)				
| φ (vi)- φ (vj)|	P_3= |2(m-n)-1|	P_5= |2(m-n)+3|				

Here, each P_i $(i = 1, 2, 3, 4, 5)$ is an odd prime. Let $2(m - n) = x \Rightarrow x$ is an even integer. So, our purpose is to find that even integer x for which each of $P_1 = |x - 3|, P_2 = |x - 5|, P_3 = |x - 1|, P_4 = |x + 1|, P_5 = |x + 3|$ is an odd prime ..... (IV)
It is obvious that $x \neq \pm 2, \pm 4, \pm 6$ otherwise at least one of $P_i = 1$ as defined in equation (IV), which contradicts the fact that each P_i is an odd prime. So, either $x > 6$ or $x < -6$. Without loss of generality, assume that $x > 6$, so the equation (IV) reduces to $P_1 = x - 3, P_2 = x - 5, P_3 = x - 1, P_4 = x + 1, P_5 = x + 3$ ..... (V)
Firstly, we claim that none of P_i can be 3 ..... (VI)
If $P_1 = 3$, then $x = 6$, which is a contradiction as $x > 6$. If $P_2 = 3$, then $x = 8 \Rightarrow P_4 = 9$, which is not a prime number. If $P_3 = 3$, then $x = 4$, which is a contradiction as $x > 6$. If $P_4 = 3$, then $x = 2$, which is again a contradiction as $x > 6$. If $P_5 = 3$, then $x = 0$ which implies taht $P_4 = 1$, which is not a prime number. So, it is clear that none of P_i can be 3. Further, equation (V) implies that $P_1 - P_2 = 2, P_3 - P_1 = 2, P_4 - P_3 = 2, P_5 - P_4 = 2$ which means that the pairs (P_1, P_2), (P_1, P_3) are twin odd primes with P_1 as common and same way (P_4, P_3), (P_4, P_5) are twin odd primes with P_4 as common. We know that each odd prime is of the form either $4k + 1$ or $4k + 3$, k being a natural number. So, if $P_1 = 4k + 1$, then $P_2 = 4k - 1, P_3 = 4k + 3, P_4 = 4k + 1, P_5 = 4k + 3$ or if $P_1 = 4k + 3$, then $P_2 = 4k + 1, P_3 = 4k + 5, P_4 = 4k + 7, P_5 = 4k + 9$, for a fixed natural numer k. So, either P_2, P_1, P_3 or P_2, P_1, P_3, P_4 , P_5 will be consecutive twin odd primes. Since every third odd number is divisble by 3, which means that no three successive odd numbers can be prime unless one of them is 3, which is impossible by assertion (VI). Hence, our supposition is wrong. So, $\varphi(v_5)$ cannot be an even integer and this case is rejected.

Case 2: When $\varphi(v_5)$ is an odd integer

Let's assume that $\varphi(v_5)$ be an odd intgeger. Since (v_6, v_5) and (v_4, v_5) are adjacent vertices in Figure 1, so as proved in Case 1, $(\varphi(v_5), \varphi(v_6))$ and $(\varphi(v_5), \varphi(v_4))$ must be pair of consecutive odd numbers and since difference of any two consecutive odd integers is 2. So, we can write their values as $\varphi(v_6) = 2n + 1, \varphi(v_5) = 2n + 3$ and $\varphi(v_4) = 2n + 5$, where n is any integer. Hence, $\Delta v_5 v_4 v_3$ implies that $\varphi(v_3)$ must be an even integer by Lemma 1. But (v_3, v_1) is an adjacent pair of vertices and $\varphi(v_1) = 2m$. So, $\varphi(v_3) = 2m - 2$. Hence, we have labeling of graph in Figure 1 as: $\varphi(v_1) = 2m, \varphi(v_2) = 2m + 2, \varphi(v_3) = 2m - 2$, $\varphi(v_4) = 2n + 5, \varphi(v_5) = 2n + 3$, and $\varphi(v_6) = 2n + 1$ since the function $\varphi: V(\Gamma(D_8)) \to \mathbb{Z}$ determines the prime distance labeling on graph shown in Figure 1. So, we have the following interpretations:

	(v_1,v_4)	(v_1,v_6)	(v_2,v_4)	(v_2,v_5)	(v_2,v_6)	(v_3,v_4)

(v_i,v_j) →						
| φ (v_i)- φ (v_j)|	P_1= |2(m-n)-5|	P_2= |2(m-n)-1|	P_3= |2(m-n)-3|	P_2= |2(m-n)-1|	P_4= |2(m-n)+1|	P_5= |2(m-n)-7|
(v_i,v_j) →	(v_3,v_5)	(v_3,v_6)				
| φ (v_i)- φ (v_j)|	P_1= |2(m-n)-5|	P_3= |2(m-n)-3|				

Here, each P_i $(i = 1,2,3,4,5)$ is an odd prime. Let $2(m-n) = x \Rightarrow x$ is an even integer. So, our purpose is to find that even integer x for which each of P_1=|x-5|, P_2 =|x-1|, P_3 =|x-3|, P_4= |x+1|, P_5 =|x-7| is an odd prime. It is obvious that $x \neq \pm 2, \pm 4, \pm 6, \pm 8$ otherwise at least one of $P_i = 1$ as defined in equation (IV), which contradicts the fact that each P_i is an odd prime. So, either $x > 8$ or $x < -8$. Without loss of generality, assume that $x > 8$. So the equation (IV) reduces to P_1=x-5, P_2=x-1, P_3=x-3, P_4=x+1, P_5 =x-7.... (VII)
Firstly, we claim that none of P_i can be 3 or 5... (VIII)
If P_1 = 3, then $x = 8$, which is a contradiction as $x > 8$. Also, if P_1 = 5, then x=10 & so P_2=9, which is not a prime number. If P_2=3 or 5, then $x = 4$ or 6 which is impossible as $x > 8$. If P_3=3 or 5, then $x = 6$ or 8, which is a contradiction as $x > 8$. If P_4= 3 or 5, then x=2 or 4, which is again a contradiction as x > 8. If P_5=3, then x= 10 or 12 which implies that P_2= 9 or P_3 =9, which is not a prime number. So, it is clear that none of P_i can be 3 or 5. Further, the equation (VII) implies that P_1-P_5= 2, P_3-P_1= 2, P_2-P_3= 2, P_4-P_2= 2 which further imply that (P_1, P_5), (P_1, P_3) and (P_2, P_3), (P_4, P_2) are pair of twin odd primes with P_1 and P_2 as common odd primes respectively. This condition is to the one we have obtained in Case 1. So, by the same argument as we applied in Case 1, this case is also rejected. Hence, $\varphi(v_5)$ can neither be an odd nor an even integer. Hence, our supposition is wrong and so we conclude that the prime distance labeling of the non-commuting graph $\Gamma(D_8)$ does not exist.

Theorem 4.
The non-commuting graph of dihedral group of order $2n$ does not permit a prime distance labeling for $n \geq 5$.

Proof.
Consider Dihedral group D_{2n} of order $2n, n \geq 5$ given by: $D_{2n} = \{< x, y >: x^n = e = y^2, xy = yx^{-1}\}$ where e is identity element of group. The centre of dihedral group Z(D_{2n}) =

$$\begin{cases} \{e\} & \text{, if n is odd natural number} \\ \left\{e, x^{\frac{n}{2}}\right\} & \text{, if n is even natural number.} \end{cases}$$

So if V denotes the vertex set for $\Gamma(D_{2n})$, then $= D_{2n} - Z(D_{2n})$, then V consists of non-cmmuting elements of group D_{2n}. Consider the set S = $\{x, y, xy, x^2y, x^3y\}$. We should note that order of $x = n$ is at least 5. So, $xy = yx^{-1} \neq yx$

$x(xy) = x^2y$, $(xy)x = yxx^{-1} = ye = y \Rightarrow x(xy) \neq (xy)x$

$x(x^2y) = x^3y$ and $(x^2y)x = yx^{-2}x = yx^{-1} = xy \neq x^3y \Rightarrow x(x^2y) \neq (x^2y)x$

$x(x^3y) = x^4y$ & $(x^3y)x = yx^{-3}x = yx^{-2} = x^2y \neq x^4y \Rightarrow x(x^3y) \neq (x^3y)x$

$y(xy) = x^{-1}yy = x^{-1}e = x^{-1}$ & $(xy)y = x \neq x^{-1} \Rightarrow y(xy) \neq (xy)y$

$y(x^2y) = yyx^{-2} = y^2x^{-2} = ex^{-2} = x^{-2}$ & $(x^2y)y = x^2y^2 = x^2e = x^2 \neq x^{-2}$
$\Rightarrow y(x^2y) \neq (x^2y)y$

$(xy)(x^2y) = x(yy)x^{-2} = xex^{-2} = x^{-1}$& $(x^2y)(xy) = x^2yyx^{-1} = x \neq x^{-1}$
$\Rightarrow (xy)(x^2y) \neq (x^2y)(xy)$

$(xy)(x^3y) = xyyx^{-3} = x^{-2}$ & $(x^3y)(xy) = x^3yyx^{-1} = x^2 \neq x^{-2} \Rightarrow (xy)(x^3y) \neq (x^3y)(xy)$

$(x^2y)(x^3y) = x^2yyx^{-3} = x^{-1}$ & $(x^3y)(x^2y) = x^3yyx^{-2} = x \neq x^{-1}$
$\Rightarrow (x^2y)(x^3y) \neq (x^3y)(x^2y)$

But $y(x^3y) = yyx^{-3} = y^2x^{-3} = ex^{-3} = x^{-3}$ & $(x^3y)y = x^3y^2 = x^3e = x^3 = x^{-3}$ if $n = 6$ ...(I)

So, (x^3y) commute with y in Dihedral group D_{12} whereas the above equations show that the pairs $(x,y),(x,xy),(x,x^2y),(x,x^3y),(y,xy),(y,x^2y),(xy,x^2y),(xy,x^3y),(x^2y,x^3y)$ are non-commuting in each D_{2n}, for $n \geq 5$. We divide the proof into the following two cases:

Case 1: When n is an odd integer

Since $n \geq 5$ and n is odd, so $n \neq 6$. By (I), $x^3 \neq x^{-3}$ and (x^3y) does not commute with y. So, in non-commuting graph $\Gamma(D_{2n})$, the following are minimum pairs of adjacent vertices: $(x,y),(x,xy),(x,x^2y),(x,x^3y),(y,xy),(y,x^2y),(xy,x^2y),(xy,x^3y),(x^2y,x^3y),(y,x^3y)$. And the set S = $\{x, y, xy, x^2y, x^3y\} \subseteq V$. Set $x = v_1, y = v_2, xy = v_3, x^2y = v_4, x^3y = v_5$. Hence, in this case the non-commuting graph $\Gamma(D_{2n})$ has the following form:

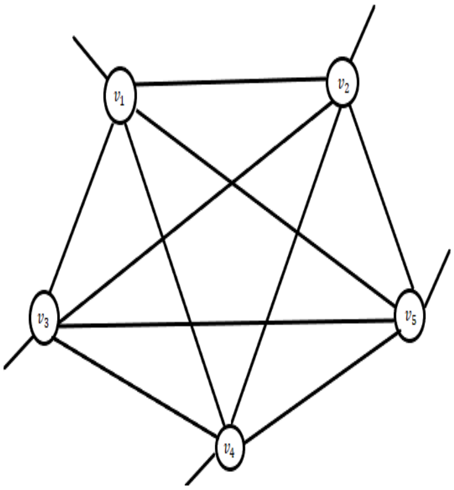

Figure 2. Subgraph of Non-Commuting graph $\Gamma(D_{2n})$

Clearly, the chromatic number of the graph shown in Figure 2 is 5. Since it is subgraph of $\Gamma(D_{2n})$, it implies that the chromatic number of $\Gamma(D_{2n})$ is greater than or equal to 5. As we know that, the prime distance labeling of any graph with chromatic number 5 is not possible. So, we conclude that for any odd number $n \geq 5$, the prime distance labeling of non-commuting graph of D_{2n}, is not possible.

Case 2: When n is an even integer

Since $n \geq 5$ and is an even integer, so n is at least 6. Also, equation (I) implies that $x^3 = x^{-3}$ if $n = 6$, so (x^3y) commutes with y in group D_{12}. So, firstly we shall prove the result for D_{12} separately. We know that $D_{12} = \{e, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y\}$. Consider the set S = $\{x, y, xy, x^2y, x^3y, x^4y, x^5y\} \subseteq V$, where V is set of vertices of $\Gamma(D_{12})$. Then, $x(x^4y) = x^5y$ & $(x^4y)x = x^4x^{-1}y = x^3y \neq x^5y \Rightarrow x(x^4y) \neq (x^4y)x$

$x(x^5y) = x^6y = ey = y$ & $(x^5y)x = x^5x^{-1}y = x^4y \neq y \Rightarrow x(x^5y) \neq (x^5y)x$

$y(x^4y) = x^{-4}yy = x^{-4} = x^2$ & $(x^4y)y = x^4 \neq x^2 \Rightarrow y(x^4y) \neq (x^4y)y$

$y(x^5y) = x^{-5}yy = x^{-5}e = x^{-5} = x$ & $(x^5y)y = x^5 \neq x \Rightarrow y(x^5y) \neq (x^5y)y$

$(x^2y)(x^4y) = x^2yyx^{-4} = x^{-2} = x^4$ & $(x^4y)(x^2y) = x^4yyx^{-2} = x^2 \neq x^4$
$\Rightarrow (x^2y)(x^4y) \neq (x^4y)(x^2y)$

$(xy)(x^5y) = xyyx^{-5} = x^{-4} = x^2$ & $(x^5y)(xy) = x^5yyx^{-1} = x^4 \neq x^2 \Rightarrow (xy)(x^5y) \neq (x^5y)(xy)$

$(x^5y)(x^4y) = x^5yyx^{-4} = x$ & $(x^4y)(x^5y) = x^4yyx^{-5} = x^{-1} = x^5 \neq x$
$\Rightarrow (x^5y)(x^4y) \neq (x^4y)(x^5y)$.

Hence, in the non-commuting graph $\Gamma(D_{12})$, the following are minimum pairs of adjacent vertices: $(x,y),(x,xy),(x,x^2y)$,

$(x, x^4y), (x, x^5y), (y, xy), (y, x^2y), (y, x^4y), (y, x^5y), (xy, x^2y), (xy\ , x^5y)$, $(x^2y, x^4y)\ , (x^5y\ , x^4y)$. By setting $x = v_1, y = v_2, xy = v_3\ , x^2y = v_4, x^4y = v_5, x^5y = v_6$, we have the following as a subgraph of $\Gamma(D_{12})$.

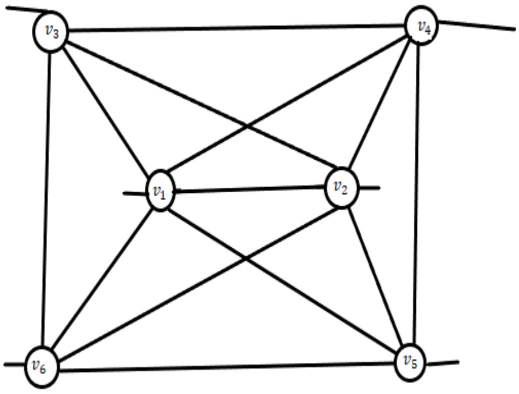

Figure 3. A Subrgraph of Non-Commmuting graph $\Gamma(D_{12})$

Let's suppose that there exist a function $\varphi: V(\ \Gamma(D_{12})) \to \mathbb{Z}$ which determines a prime distance labeling such that $|\varphi(u) - \varphi(v)|$ is an odd prime for each pair (u, v) of adjacent verices in graph $\Gamma(D_{12})$. Then it is clear from Figure 3 and by Lemma 1, that in $\Delta v_1v_2v_3$, for any two of three vertices should be labeled with either an odd or even integer. Assuming $\varphi(v_1) = 2n, \varphi(v_2) = 2n + 2, n \in \mathbb{Z}$, then $\varphi(v_3)$ must be odd integer, say $\varphi(v_3) = 2m + 1, m \in \mathbb{Z}$ ... (II)

So, from $\Delta v_1v_2v_4$, we get that $\varphi(v_4)$ must be an odd integer and also (v_3, v_4) are adjacent vertices which imply that $\varphi(v_4) = 2m + 3$ ... (III)

Proceeding in similar way, from $\Delta v_1v_2v_6$ and $\Delta v_1v_2v_5$, using the fact that $(v_3, v_6), (v_5, v_4)$ are adjacent vertices, we get $\varphi(v_6)$ and $\varphi(v_5)$ must be odd integers and so, $|\varphi(v_6) - \varphi(v_3)|$ and $|\varphi(v_4) - \varphi(v_5)|$ are even prime numbers. It implies that $(\varphi(v_6), \varphi(v_3))$ and $(\varphi(v_4), \varphi(v_5))$ must be pair of consecutive odd integers ... (IV)

But (v_5, v_6) is a pair of adjacent vertices. So, $|\varphi(v_6) - \varphi(v_5)|$ will be even prime number (difference of any two odd integers is even) if and only if $\varphi(v_6)$ and $\varphi(v_5)$ are consecutive odd integers ... (V)

Combining statements (II), (III), (IV), & (V), we see that $(\ \varphi(v_6), \varphi(v_3)\ ), \big(\varphi(v_4)\ , \varphi(v_5)\big), ((\varphi(v_6), \varphi(v_5)\ )$, and $(\varphi(v_4), \varphi(v_3))$ are pairs of consecutive odd integers. Again (II) and (III) imply that, $\varphi(v_5)$= $2m + 5$, $\varphi(v_6) = 2m - 1$ or $2m + 7$. So, $|\varphi(v_6) - \varphi(v_5)| = 6$ or $|\varphi(v_6) - \varphi(v_3)| = 6$, a contradiction. Hence, it is not possible to label the subgraph as shown in Figure 3 with function $\varphi: V(\ \Gamma(D_{12})) \to \mathbb{Z}$ such that $|\varphi(u) - \varphi(v)|$ is an odd prime for each pair (u, v) of adjacent verices in Figure 3. So, if a subgraph of $\Gamma(D_{12})$ does not admit a prime distance labeling, then $\Gamma(D_{12})$ will not possess a prime distance labeling. Hence, the prime distance labeling of the non-commuting grpah of D_{2n}, for $n = 6$ is not possible. Further, we are left to prove the statement for even integers $n > 6$. $\therefore n \geq 10$ & n is even integer. So equation (I) implies that $y(x^3y) = yyx^{-3} = y^2x^{-3} = ex^{-3} = x^{-3}$ & $(x^3y)y = x^3y^2 = x^3e = x^3 \neq x^{-3}$ for $n = 10,12\ldots$ So, (x^3y) commute with y in Dihedral Group D_{2n} for even integers $n \geq 10$. Hence, $(x, y), (x, xy), (x, x^2y)\ , (x, x^3y), (y, xy), (y, x^2y), (xy, x^2y), (xy\ , x^3y), (x^2y, x^3y), (y, x^3y)$ are non-commuting in each D_{2n}, for even integers $n \geq 10$ and the set S = $\{\ x, y\ , xy\ , x^2y\ , x^3y\} \subseteq V$. Set $x = v_1, y = v_2, xy = v_3\ , x^2y = v_4, x^3y = v_5$. Hence, in this case the non-commuting graph $\Gamma(D_{2n})$ for even integers $n \geq 10$ has the following form:

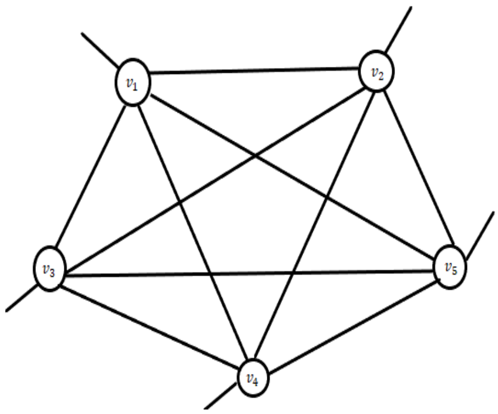

Figure 4. A Subgraph of $\Gamma(D_{2n})$ for even integers $n \geq 10$

This is similar to Case 1. As stated in Case 1, the chromatic number of the graph shown in Figure 4 is 5. Since it is a subgraph of $\Gamma(D_{2n})$, for even integers $n \geq 10$, it implies that the chromatic number of $\Gamma(D_{2n})$ is greater than or equal to 5. As we know that, the prime distance labeling of any graph with chromatic number 5 is not possible. So, we conclude that for any even integers $n \geq 10$, the prime distance labelling of the non-commuting graph of D_{2n}, is not possible. Hence, combining both Cases 1 & 2, we conclude the prime distance labelling of the non-cmmuting graph of Dihedral goups D_{2n}, for $n \geq 4$, does not exist.

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