ANALYZATION OF A FUZZY QUEUING MODEL WITH FLEXIBLE SERVICE POLICY USING PARAMETRIC PROGRAMMING APPROACH

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Abstract:

Fuzziness is a type of new incoherence. The fuzzy set theory is said to depict ambiguity.Fuzzy queuing model with flexible service policy is being investigated in this work. A parametric programming technique is designed to find the membership functions of queue length and sojourn time in steady state, in which the arrival rate and service rate are being pentagon fuzzy numbers. Based on α-cut approach and Zadeh's extension principle, the fuzzy queues are transformed into family of crisp queues. The model's potency is calculated for various possibilities of α-cuts.

Keywords:

Fuzzy Sets, Membership Functions, Fuzzy Queuing Model, Pentagon fuzzy number,Flexible Service Policy,Parametric Programming, α-cut,Queue length, Sojourn time,

1. Introduction:

Queuing Theory is an important facet of operational research. Applications of waiting lineis very essential in day to day life but also in sequence of computer programming, networks, medical field, banking sectors, call centres, telecommunications, manufacturing and production systems.

Many researchers have investigated the optimization problems of queuing systems, as well as their fine structure and techniques in several Markovian lines. Various service policies bring different functionality to multi-server queue systems. Burnetas and Economou[1] was to propose anequilibrium strategies of customers in several Markovian queues. Many scholars have designed and contributed Markovian queues in diverse contexts, includingEconomou and Kanta[2], Guo and Hassin[9], Li and Li[8], Lan and Tang. Multi server queuing systems with different service policies has been analysed by the researchers like T. V. Do, Baumann and Sandmann, Liu and Yu[11].

Fuzzy queuing model have been described by the authors likeR.J. Li and E.S. Lee [10], J.J. Buckley[7], R.S. Negi and E.S. Lee[4].Chen has discussed the queuing system in a fuzzy environment using Zadeh's extension principle. Kao et al derived the membership functions of the system measures for fuzzy queues using parametric programming technique.

This paper is structuredas follows: second segment provides some basic definitions of this research work. Third segment details a mathematical model which figures queue length and sojourn time.Fourth segment describes the parametric programming technique of a multi-server fuzzy queue with flexible service policy. Fifth segment illustrates a numerical example. Sixth segment concludes the paper.

2. Preliminaries:

Definition: 2.1

A is a fuzzy set defined on E and can be written as a collection of ordered pairs. If E is a universe of discourse and x is a particular element of E, then $\tilde{A}=\{(x, \phi_{\tilde{A}}(x)) / x \in E\}$.

Definition: 2.2

The α - cut of a fuzzy number $\tilde{A}$ is defined as $\tilde{A}=\{x : \phi_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1]\}$

Definition: 2.3

A Trapezoidal fuzzy number is defined as $\tilde{A}=(a,b,c,d)$ where a < b < c < d withits membership function is given by:

$$\phi_{\tilde{A}}(x)=\begin{cases} L(x)=\dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1 & , b \leq x \leq c \\ R(x)=\dfrac{d-x}{d-c}, & c \leq x \leq d \\ 0 & , otherwise \end{cases}$$

Definition: 2.4

A Pentagon fuzzy number is defined as $\tilde{A}=(a_1,a_2,a_3,a_4,a_5)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ withits membership function is given by:

$$\phi_{\tilde{A}}(x)=1\begin{cases} 0 & , x < a_1 \\ L_1(x)=\dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ L_2(x)=\dfrac{x-a_2}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 1 & , x = a_3 \\ R_1(x)=\dfrac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ R_2(x)=\dfrac{a_5-x}{a_5-a_4}, & a_4 \leq x \leq a_5 \\ 0 & , x > a_5 \end{cases}$$

Definition: 2.5

Zadeh's Extension Principle:

Let the inter arrival rate $\tilde{\lambda}$ and service rate $\tilde{\mu}$ are all fuzzy numbers. The system performance measure $P(x, y)$ is defined as

$$\phi_{p(\tilde{A},\tilde{S})}(Z) = Sup\left\{\min(\phi_{\tilde{A}}(x), \phi_{\tilde{S}}(y)) / Z = p(x, y)\right\}$$

$x \in X$

$y \in Y$

3. Model Description:

We consider the M/M/2 Queuing system with an infinite waiting room in a fuzzy environment with flexible service policy. The customers arrive according to a Poisson process with fuzzy rate $\tilde{\lambda}$.When the system has at least two customers, each server provides service to one of them separately. When the system only has one customer, however, the two servers serve that customer collectively at the same time.It is called the TTO service policy.

If the system has two identical servers, but one server can serve two customers at the same time. If the system has at least two customers, each server services a customer individually, or if the system only has one customer, both servers serve the customer simultaneously. If another customer arrives before servicing a customer, one of the two servers turns to service the new customer immediately. When a customer is serviced by one server, the service time is an exponential distribution with the parameter $\tilde{\mu}$. On the other hand, when a customer is serviced by two servers together, we introduce the interaction parameter q.The traffic intensity $\tilde{\rho}=\frac{\tilde{\lambda}}{2\tilde{\mu}}$

The balance equations governing the fuzzy queuing model are derived as follows:

$$\tilde{\lambda}P_0 - 2\tilde{\mu}qP_1 = 0,$$
$$\tilde{\lambda}P_0 - (\tilde{\lambda} + 2\tilde{\mu}q)P_1 + 2\tilde{\mu}P_2 = 0,$$
$$\tilde{\lambda}P_i - (\tilde{\lambda} + 2\tilde{\mu})P_{i+1} + 2\tilde{\mu}P_{i+2} = 0, \qquad i \geq 1$$

Solving the above equations, we obtain:

$$P_0 = \Pr\{\text{system is empty}\}$$
$$= \frac{2\tilde{\mu}q - \tilde{\lambda}q}{2\tilde{\mu}q - \tilde{\lambda}q + \tilde{\lambda}}$$
$$P_i = \left(\frac{\tilde{\lambda}}{2\tilde{\mu}}\right)^i \frac{2\tilde{\mu} - \tilde{\lambda}}{2\tilde{\mu}q - \tilde{\lambda}q + \tilde{\lambda}}, \qquad i \geq 1$$

$$P_{NW} = \Pr\{\text{the customer need not to wait}\}$$

$$= P_0 + P_1$$
$$= \frac{4\tilde{\mu}^2 q - 2\tilde{\lambda}\tilde{\mu}q + 2\tilde{\mu}\tilde{\lambda} - \tilde{\lambda}^2}{4\tilde{\mu}^2 q - 2\tilde{\mu}\tilde{\lambda}q + 2\tilde{\mu}\tilde{\lambda}},$$

3.1. Performance Measures:

(i) Steady-state queue Length $\left(N_q\right)$

$$N_q = \frac{2\tilde{\mu}\tilde{\lambda}}{[(2\tilde{\mu} - \tilde{\lambda})q + \tilde{\lambda}](2\tilde{\mu} - \tilde{\lambda})},$$

(ii) Steady-stateSojourn time $\left(W_S\right)$

$$W_s = \frac{2\tilde{\mu}}{[(2\tilde{\mu} - \tilde{\lambda})q + \tilde{\lambda}](2\tilde{\mu} - \tilde{\lambda})},$$

3.2. Remark:

Letting **q**=0.5, the system can be turned into classical FM/FM/2 queuing system.

4. Parametric Programming Approach of a Fuzzy Queuing Model:

Consider a two server fuzzy queue system with flexible service policy. The inter arrival time $\tilde{A}$ and service time $\tilde{S}$ are denoted by the following fuzzy sets

$$\tilde{A} = \left\{(a, \phi_{\tilde{A}}(a)) / a \in X\right\}$$

$$\tilde{S} = \left\{(s, \phi_{\tilde{S}}(s)) / s \in Y\right\}$$

Where X and Y are crisp sets of inter arrival time and service time.Using α -cut, the arrival rate and service rate can be represented by various levels of confidence intervals [0,1]. Let the confidence intervals of fuzzy sets $\tilde{A}$ and $\tilde{S}$ be $\left[l_{A(\alpha)}, u_{A(\alpha)}\right]$ and $\left[l_{S(\alpha)}, u_{S(\alpha)}\right]$.Thus the crisp queue can be got from the fuzzy queue for different α -cuts. The membership function of the performance measures $p(\tilde{A}, \tilde{S})$ is defined as

$$\phi_{p(a,s)}(Z) = Sup\left\{\min(\phi_{\tilde{A}}(a), \phi_{\tilde{S}}(s)) / Z = p(a,s)\right\}$$

$$a \in X$$

$$s \in Y$$

The parametric programming technique is used for finding upper and lower bounds of α -cuts of $\phi_{p(a,s)}(Z)$ which are $l_{p(\alpha)} = \min p(a,s)$; Such that $l_{A(\alpha)} \leq a \leq u_{A(\alpha)}$ and $l_{S(\alpha)} \leq s \leq u_{S(\alpha)}$

$u_{p(\alpha)} = \max p(a,s)$;Such that $l_{A(\alpha)} \leq a \leq u_{A(\alpha)}$ and $l_{S(\alpha)} \leq s \leq u_{S(\alpha)}$

Both the upper and lower bounds of p(α) is invertible with respect to α . The left shape Function L(z) and right shape function R(z) can be obtained from $l_{p(\alpha)}^{-1}$ and $u_{p(\alpha)}^{-1}$.

$$\phi_{P(a,s)}(z) = \begin{cases} L_1(z), & z_1 \leq z \leq z_2 \\ L_2(z), & z_2 \leq z \leq z_3 \\ 1, & z = z_3 \\ R_1(z), & z_3 \leq z \leq z_4 \\ R_2(z), & z_4 \leq z \leq z_5 \end{cases}$$

Where $z_1 \leq z_2 \leq z_3 \leq z_4 \leq z_5$ and $L_1(z) = R_2(z) = 0$ for pentagon fuzzy number.

5. Numerical Illustration:

Consider a two server fuzzy queuing system that runs on a two-to-one(TTO) servicepolicy basis.The service rate is assumed to follow exponential distribution, whereas the arrival rate follows the Poisson distribution. When a customer is serviced by two servers together, we introduce the interaction parameter q. In this system, we take q=1. A parametric programming technique can be used to evaluate the efficiency of the fuzzy queuing model.

Let the inter arrival rate and service rate are Pentagon fuzzy numbers. Let $\tilde{\lambda}=[1,2,3,4,5]$ and $\tilde{\mu}=[11,12,13,14,15]$.

The α -cut of their membership functions are $[1+2\alpha,\, 5-2\alpha],[11+2\alpha,\, 15-2\alpha]$.

(i) Steady-State queue Length (N_q)

$$N_q = \frac{2\tilde{\mu}\tilde{\lambda}}{[(2\tilde{\mu}-\tilde{\lambda})q+\tilde{\lambda}](2\tilde{\mu}-\tilde{\lambda})},$$

The parametric programming for the steady state queue length is

$$l_{N_q(\alpha)} = Min\left\{\frac{2xy}{[(2y-x)+x](2y-x)}\right\}, \text{Such that } 1+2\alpha<x<5-2\alpha \text{ and } 11+2\alpha<y<15-2\alpha$$

$$u_{N_q(\alpha)} = Max\left\{\frac{2xy}{[(2y-x)+x](2y-x)}\right\}, \text{ Such that } 1+2\alpha<x<5-2\alpha \text{ and } 11+2\alpha<y<15-2\alpha .$$

For $l_{N_q(\alpha)}, x \to 1+2\alpha \text{ and } y \to 15-2\alpha$

$$l_{N_q(\alpha)} = \left\{\frac{-8\alpha^2+56\alpha+30}{24\alpha^2-296\alpha+870}\right\}$$

For $u_{N_q(\alpha)}, x \to 5-2\alpha \text{ and } y \to 11+2\alpha$

$$u_{N_q(\alpha)} = \left\{\frac{-8\alpha^2-24\alpha+110}{24\alpha^2+200\alpha+374}\right\}$$

The membership function

$$\phi_{N_q}(z) = \begin{cases} L(z), & \left[l_{N_q(\alpha)}\right]_{\alpha=0} \le z \le \left[l_{N_q(\alpha)}\right]_{\alpha=1} \\ R(z), & \left[u_{N_q(\alpha)}\right]_{\alpha=1} \le z \le \left[u_{N_q(\alpha)}\right]_{\alpha=0} \\ 0 & , \text{otherwise} \end{cases}, \text{ Which is defined as}$$

$$\phi_{N_q}(z) = \begin{cases} \dfrac{(296z+56)\pm 64\sqrt{(z^2+2z+1)}}{48z+16} & ,0.0345 \le z \le 0.1304 \\ \dfrac{-(200z+24)\pm 64\sqrt{(z^2+2z+1)}}{48z+16} & , 0.1304 \le z \le 0.2941 \\ 0 & , \text{otherwise} \end{cases}$$

(ii) Steady-State Sojourn time (W_S)

$$W_s = \frac{2\tilde{\mu}}{[(2\tilde{\mu}-\tilde{\lambda})q+\tilde{\lambda}](2\tilde{\mu}-\tilde{\lambda})},$$

The parametric programming for the steady-state Sojourn time is

$$l_{W_s(\alpha)} = Min\left\{\frac{2y}{[(2y-x)+x](2y-x)}\right\}, \text{Such that } 1+2\alpha<x<5-2\alpha \text{ and } 11+2\alpha<y<15-2\alpha .$$

$u_{W_s(\alpha)} = Max\left\{\frac{2y}{[(2y-x)+x](2y-x)}\right\}$, Such that $1+2\alpha < x < 5-2\alpha$ *and* $11+2\alpha < y < 15-2\alpha$.

For $l_{W_s(\alpha)}, x \to 1+2\alpha$ *and* $y \to 15-2\alpha$

$$l_{W_s(\alpha)} = \left\{\frac{30-4\alpha}{24\alpha^2-296\alpha+870}\right\}$$

For $u_{W_s(\alpha)}, x \to 5-2\alpha$ *and* $y \to 11+2\alpha$

$$u_{W_s(\alpha)} = \left\{\frac{22+4\alpha}{24\alpha^2+200\alpha+374}\right\}$$

The membership function

$$\phi_{W_s}(z) = \begin{cases} L(z), & \left[l_{W_s(\alpha)}\right]_{\alpha=0} \le z \le \left[l_{W_s(\alpha)}\right]_{\alpha=1} \\ R(z), & \left[u_{W_s(\alpha)}\right]_{\alpha=1} \le z \le \left[u_{W_s(\alpha)}\right]_{\alpha=0} \\ 0 & , otherwise \end{cases}$$

Which is defined as

$$\phi_{W_s}(z) = \begin{cases} \dfrac{(296z-4) \pm 4\sqrt{(256z^2+32z+1)}}{48z} & , 0.0345 \le z \le 0.04345 \\ \dfrac{-(200z-4) \pm 4\sqrt{(256z^2+32z+1)}}{48z} & , 0.0435 \le z \le 0.0588 \\ 0 & , otherwise \end{cases}$$

Table1. α -cuts of queue length, sojourn time

S. No.	α	$l_{N_{q\alpha}}$	$u_{N_{q\alpha}}$	$l_{W_{s\alpha}}$	$u_{W_{s\alpha}}$
1	0	0.0345	0.2941	0.0345	0.0588
2	0.1	0.0423	0.2727	0.0352	0.0568
3	0.2	0.0504	0.2527	0.0360	0.0549
4	0.3	0.0588	0.2340	0.0368	0.0532
5	0.4	0.0677	0.2165	0.0376	0.0515
6	0.5	0.0769	0.2000	0.0385	0.0500
7	0.6	0.0866	0.1845	0.0394	0.0485
8	0.7	0.0968	0.1698	0.0403	0.0472
9	0.8	0.1074	0.1560	0.0413	0.0459

10	0.9	0.1186	0.1429	0.0424	0.0446
11	1	0.1304	0.1304	0.0435	0.0435

6. Conclusion

This Paper investigates the qualitative behavior of two server fuzzy queuing systems with flexible service policies using the pentagon fuzzy number. Using the α-cut and Zadeh's extension principle, we determine the queue length and sojourn time in steady-state. Thus, the example has successfully demonstrated the effectiveness and precision of parametric programming.

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