# Generalized Form Of Closure Set For Topological Space

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Abstract: We are going to define the new general form of closure set for any subset of the topological space and study their properties. Furthermore, we also establish relationship of new defined form with existing form.

#### Introduction

For the first time, in 1970 N. Levine studied about new general form of closed set called g – closed set in [1]. As we know that topology on a set is defined in term of open set and complementary of each open set is closed set. Several properties of topological space studied in term of closed set like continuity, conecctededness, compactness, countability, separation axiom. Levine and many reserachers studied several properties of topological space in term of this newly defined generalized form of closed set and found very singnficant reuslts. In continuoution of such study, researchers also intiated to defined other general forms of closed sets which help to understand many more properties of topological space in [2-16]. The reder can study some important generalization of closed set as generalized pre-regular weakly closed set (in short *gprw*- closed set) and study thier properties. We are going to define new general form of closure set as called *gprw*-closure set (shortly dentoed by  $\beta C$  –closure set) which is actually based of *gprw*-closed set. Furthermore, we investigate properties of  $\beta C$  –closure set.

First, we look at definitions and results for better understanding about the main results. These are the notation and symbols used throughout the paper. Here C(A) and I(A) are notation of closure and interior of any subset A of space X respectively. The space means a topological space  $(X, \tau)$  (or X) otherwise we will mention. The gprw-set means generalized pre-regular weakly set. We use symbol  $\beta$  instead of gprw, for example gprw-closure set means  $\beta$ -closure set.

If  $A \subset X$ , then A is

i. g-closed set  $\Leftrightarrow C(A) \subseteq B$ , when  $A \subseteq B$ , here B is open in X.

- ii. r-open (or r-closed) if A = I(C(A)) or A = C(I(A)).
- iii. pre-open (or pre-closed) if  $A \subseteq I(\mathcal{C}(A))$  or  $\mathcal{C}(I(A)) \subseteq A$ .
- iv. regular semiopen if  $\exists r \text{open set } U \text{ s.t. } U \subset A \subset C(U)$ .
- v.  $\beta$  -closed if  $pC(A) \subset U$ , if A lies in regular semi open set U.
- vi. rw closed if  $C(A) \subseteq U$  if A lies in (or equal) to regular semi open set U.

#### **Main Result**

Now we define new general form of closure set as  $\beta$  –closure set for any subset of space X and study their properties. Furthermore, we also find the relationship with other general form

of closed sets.

The  $\beta$ -closure of any subset U of space X is denoted by  $\beta C(U)$  and it the intersection of all  $\beta$ -closed set which containing U i.e.  $\beta C(U) = \bigcap \{F : U \subset F, where F \text{ is } \beta - \text{closed} \}$ .

**Theorem 1** For a space X,  $\beta C(X) = X$  and  $\beta C(\phi) = \phi$ .

*Proof.* Due to definition of  $\beta$ -closure, X is the only  $\beta$ -closed lies in . Hence,  $\beta C(X)$  is equal to intersection of all the  $\beta$ -closed set lies in X i.e.  $= \cap \{X\} = X$ . That is  $\beta C(X) = X$ . Also due to the definition of  $\beta$ -closure,  $\beta C(\phi)$  is equal to the intersection of all the  $\beta$ -closed sets containing  $\phi$  that is  $\phi \cap$  any gprw-closed set lies in  $\phi$  is only  $\phi$ . Hence  $\beta C(\phi) = \phi$ .

**Theorem 2** For subset A of space  $X, A \subset \beta C(A)$ .

*Proof.* By the definition of  $\beta$ -closure that is the intersection of all  $\beta$ -closed sets lies in A, so it is quite obvious that A in contained in  $\beta C(A)$ .

**Theorem 3** For two subsets A and B of space X, where B is  $\beta$ -closed set s.t.  $A \subset B$ , then  $\beta C(A) \subset B$ .

*Proof.* As given that B is  $\beta$ -closed set lies in A  $\beta C(A)$  is the intersection of all  $\beta$ -closed sets lies in A. So,  $\beta C(A)$  is lies in all  $\beta$ -closed set lies in A. Therefore, in particular  $\beta C(A) \subset B$ .

**Theorem 4** For two subsets A and B of space X if  $A \subset B$ , then  $\beta C(A) \subset \beta C(B)$ .

*Proof.* As given that A and B be subsets of X and  $A \subset B$  so,  $\beta C(B) = \bigcap \{F: B \subset F \in GPRWC(X)\}$ , now when  $B \subset F \in GPRWC(X)$ , then  $\beta C(B) \subset F$ . As,  $\subset B, A \subset B \subset F \in GPRWC(X)$ , also we have  $\beta C(A) \subset F$ . Therefore  $\beta C(A) \subset \bigcap \{F: B \subset F \in GPRWC(X)\} = gprw - cl(B)$ . Here GPRWC(X) means the collection of all  $\beta$  - closed set for space X.

**Theorem 5** For any subset A of space x,  $\beta C(A) = \beta C(\beta C(A))$ 

*Proof.* As given that A be a subset of X and we have  $\beta C(A) = \bigcap \{F : A \subset F \in GPRWC(X)\}$ . When  $A \subset F \in GPRWC(X)$ . Then  $C(A) \subset F$ . Since F is  $\beta$ -closed set containing  $\beta C(A)$ , by standard result  $\beta C(\beta C(A)) \subset F$ . Now  $\beta C(\beta C(A)) \subset \bigcap \{F : A \subset F \in GPRWC(X)\} = \beta C(A)$ . Hence  $\beta C(\beta C(A)) = \beta C(A)$ .

**Theorem 6** For a  $\beta$  -closed subset A of X,  $\beta C(A) = A$ .

*Proof.* As given that A be a  $\beta$  –-closed subset of X and since  $A \subset \beta C(A)$ . Also  $A \subset A$  and A is  $\beta$ -closed. By standard result  $\beta C(A) \subset A$ . Hence  $\beta C(A) = A$ .

Conversely of the above result is false, for example.

**Example 7** If  $X = \{1,2,3,4\}$  be a space where topology is  $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$ . The collection of all  $\beta$ -closed sets is  $\{\phi, X, \{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}\}$ . If we take  $A = \{1\}$  then  $gprw - cl(\{1\}) = \{1\}$ . But  $A = \{1\}$  is not gprw-closed, since collections of all pre-closed and regular semi open sets is  $\{\phi, X, \{3\}, \{4\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}\}$ , and  $\{\phi, X, \{1\}, \{2\}, \{2,3\}, \{1,3\}, \{2,4\}, \{1,4\}, \{2,3,4\}, \{1,3,4\}\}$ . Now we have regular semi open set  $\{1\}$  containing A, but  $pcl(A = \{1\}) = \{1,3,4\}$  is not contained in  $\{1\}$ .

**Theorem 8** For two subsets A and B of space X,  $\beta C(A \cap B) \subset \beta C(A) \cap \beta C(B)$ .

*Proof:* As given that A and B are two subsets of space X, as we know that  $A \cap B \subset A$  and  $A \cap B \subset B$  this shows that  $\beta C(A \cap B) \subset \beta C(A)$  and  $\beta C(A \cap B) \subset \beta C(B)$ . Hence  $\beta C(A \cap B) \subset \beta C(A) \cap \beta C(B)$ .

**Theorem 9** If PC(X) is closed under finite union where PC(X) be the family of all pre closed set in X, then the family of all generalized pre regular weakly closed in X denoted as GPRWC(X) is closed under finite union.

*Proof.* PC(X) is given to be closed under finite union, and let A and B belong to GPRWC(X) and  $A \cup B \subset U$ , here U is regular semi open in X, so  $A \subset U$  and  $B \subset U$ .

Therefore  $pcl(A) \subset U$  and  $pcl(B) \subset U$ , this shows that  $pcl(A) \cup pcl(B) \subset U$ . By hypothesis  $pcl(A \cup B) \subset U$ . That means  $A \cup B$  is also  $\beta$ -closed set, so we can conclude that GPRWC(X) is closed under finite union.

**Theorem 10** For two subsets A and B of space X,  $\beta C(A \cup B) = \beta C(A) \cup \beta C(B)$ .

Proof. As given that A and B are two subsets of space X, clearly  $A \subset A \cup B$  and also  $B \subset A \cup B$ . So we have  $\beta C(A) \subset \beta C(A \cup B)$  and  $\beta C(B) \subset \beta C(A \cup B)$ . Hence we got  $\beta C(A) \cup \beta C(B) \subset \beta C(A \cup B)$ . Now we shall prove the reverse inequality so in that context let  $x \in \beta C(A \cup B)$  and suppose that x does not belong to  $\beta C(A) \cup \beta C(B)$ , then there exist  $\beta$  -closed set  $A_1$  and  $B_1$  with  $A \subset A_1$ ,  $B \subset B_1$  and  $x \in A_1 \cup B_1$  so we have  $A \cup B \subset A_1 \cup B_1$  and since the space is given to be closed under finite union therefore  $A_1 \cup B_1$  is gprw-closed set by  $x \in A_1 \cup B_1$ . Thus x doses not belong to  $\beta C(A \cup B)$ . Which is a contradiction to  $x \in \beta C(A \cup B)$ . So we got  $\beta C(A \cup B) \subset \beta C(A) \cup \beta C(B)$ . Hence  $\beta C(A \cup B) = \beta C(A) \cup \beta C(B)$ .

**Theorem 11** For subset A of space  $X, \beta C(A) \subset C(A)$ .

*Proof.* Since  $A \subset X$ , so  $C(A) = \bigcap \{F \subset X : A \subset F \in C(X)\}$  where C(X) be the family all closed sets in X. If  $A \subset F \in C(X)$  then  $A \subset F \in GPRWC(X)$ . Beacuse all closed set is  $\beta$  – closed set. That is  $\beta C(A) \subset F$ . So,  $\beta C(A) \subset \bigcap \{F \subset X : A \subset F \in C(X)\} = cl(A)$ . Therefore,  $\beta C(A) \subset C(A)$ .

Remark 12 Reverse inequality does not hold in the above theorem, for example.

**Example 13** The set  $X = \{1,2,3,4\}$  space with  $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$ . Now if we take  $A = \{1\}$  then  $\beta C(A) = \{1\}$ , but  $cl(A) = \{1,3,4\}$ . Hence we can see that cl(A) is not contained in  $\beta C(A)$ .

**Theorem 14** For subset A of space  $X, \beta C(A) \subset pcl(A)$ . where pcl(A) is given by  $pcl(A) = \bigcap \{ F \subset X : A \subset F \in PC(X) \}$  where PC(X) is the collection of every pre closed sets in X.

*Proof.* Since  $A \subset X$ , so,  $pcl(A) = \bigcap \{ F \subset X : A \subset F \in PC(X) \}$  where PC(X) be the family all pre-closed sets in X. If  $A \subset F \in PC(X)$  then  $A \subset F \in GPRWC(X)$ . Since all pre-closed set is  $\beta$ -closed set. That is  $\beta C(A) \subset F$ . So,  $\beta C(A) \subset \bigcap \{ F \subset X : A \subset F \in PC(X) \} = pcl(A)$ . Therefore,  $\beta C(A) \subset pcl(A)$ .

**Theorem 15** For subset A of space  $X, \beta C(A) \subset w - C(A)$ , here w - C(A) is defined as  $\bigcap \{ F \subset X : A \subset F \in WC(X) \}$  where WC(X) is the collection of all w -closed sets in X.

*Proof.* Since  $A \subset X$ , so, we have  $w - C(A) = \bigcap \{F \subset X : A \subset F \in WC(X)\}$ . If  $A \subset F$  and F is w-closed subset of X, then  $A \subset F \in GPRWC(X)$ , since all w-closed set is  $\beta$ -closed set in X. That is  $\beta C(A) \subset F$ . So,  $\beta C(A) \subset \bigcap \{F \subset X : A \subset F \in WC(X)\} = w - C(A)$ . Therefore,  $\beta C(A) \subset w - C(A)$ , for an arbitrary subset A of topological space X.

**Remark 16** *Reverse inequality in the above theorem does not hold, for example.* 

**Example** The set  $X = \{1,2,3\}$  is space where topology is  $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$  and if we take  $A = \{1\}$  then  $w - cl(A) = \{1,3\}$ , but  $\beta C(A) = \{1\}$ . Hence w-closure is not contained in  $\beta$ -closure.

**Theorem 18** For subset A of space X,  $\beta C(A) \subset rw - C(A)$  here rw - C(A) is given by  $rw - C(A) = \bigcap \{ F \subset X : A \subset F \in RWC(X) \}$ , where RWC(X) is the family of all regular w -closed sets in X.

*Proof.* Since  $A \subset X$ , and *rw*-closure set tell us that,  $rw - C(A) = \bigcap \{F \subset X: A \subset F \in RWC(X)\}$ . If  $A \subset F$  and F is *rw*-closed subset of X, then  $A \subset F \in GPRWC(X)$ , since all *rw*-closed set is  $\beta$ -closed set in X. That is  $\beta C(A) \subset F$ . So,  $\beta C(A) \subset \cap \{F \subset X: A \subset F \in RWC(X)\} = rw - CA$ . Hence  $\beta C(A) \subset rw - C(A)$ , for an arbitrary subset A of topological space X.

**Theorem 19** For subset A of space  $X, \beta C(A) \subset rcl(A)$  where rcl(A) is given by  $rcl(A) = \bigcap \{F \subset X : A \subset F \in RC(X)\}$  where RC(X) is the family of all r -closed sets in X. Proof. Since  $A \subset X$  and regular closure set tell us that,  $rC(A) = \bigcap \{F \subset X : A \subset F \in RC(X)\}$ . If  $A \subset F$  and F is regular closed subset of X, then  $A \subset F \in GPRWC(X)$ , since all regular closed set is  $\beta$ -closed set in X. That is $\beta(A) \subset F$ . So,  $\beta C(A) \subset \bigcap \{F \subset X : A \subset F \in C(X)\}$  = rC(X) = rC(X) = rC(X).

**Theorem 20** For subset A of space X,  $\beta C(A) \subset \Pi - C(A)$  where  $\Pi - C(A)$  is given by  $\Pi - C(A) = \bigcap \{F \subset X : A \subset F \in \Pi C(X)\}$  where  $\Pi C(X)$  is the collection of all  $\Pi$ -closed sets in X.

*Proof.* Since  $A \subset X$  and  $\Pi$ -closure tell us that,  $\Pi - C(A) = \bigcap \{F \subset X : A \subset F \in \Pi C(X)\}$ . If  $A \subset F$  and F is  $\Pi$ -closed subset of X, then  $A \subset F \in GPRWC(X)$ , since all  $\Pi$ -closed set is  $\beta$ -closed set in X. That is  $\beta C(A) \subset F$ . So,  $\beta C(A) \subset \bigcap \{F \subset X : A \subset F \in \Pi C(X)\} = \Pi - cl(A)$ .

**Theorem 21** For subset A of space X,  $gpr - C(A) \subset \beta C(A)$  here gpr - C(A) is defined as  $gpr - C(A) = \bigcap \{F \subset X : A \subset F \in GPRC(X)\}$  where GPRC(X) is the family of all generalized p- regular closed sets in X.

*Proof.* Since  $A \subset X$  and  $\beta$ -closure set tell us that,  $\beta C(A) = \bigcap \{F \subset X : A \subset F \in GPRWC(X)\}$ . If  $A \subset F$  and F is  $\beta$ -closed subset of X, then  $A \subset F \in GPRC(X)$ , since all  $\beta$ -closed set is gpr-closed set in X, i.e.  $gpr - C(A) \subset F$ . So,  $gpr - C(A) \subset \bigcap \{F \subset X : A \subset F \in GPRWC(X)\} = \beta C(A)$ . Hence  $gpr - C(A) \subset \beta C(A)$ , for an arbitrary subset A of topological space X.

**Theorem 22** For any  $x \in X$ , and also  $x \in \beta C(A)$  where A is subset of a topological space X iff  $V \cap A \neq \phi$ ,  $\forall \beta$ -open set V containing x.

*Proof.* Assume that  $x \in X$ , and also  $x \in \beta C(A)$  where A is subset of a topological space X, then we show that  $V \cap A \neq \phi$ ,  $\forall \beta$ -open set V containing x. Suppose there exist  $\beta$ -open set V containing x s.t.  $V \cap A = \phi$ . Then  $A \subset X \setminus V$  and  $X \setminus V$  is  $\beta$ -closed set. We have  $\beta C(A) \subset X \setminus V$ . This shows  $x \subset \beta C(A)$ , which is a contradict or assumption, hence  $V \cap A \neq \phi$  for every  $\beta$ -open set V containing x. Conversely, when  $V \cap A \neq \phi$ ,  $\forall \beta$ -open set V containing x. To thow that  $x \in \beta C(A)$ , suppose x does not belong to  $\beta C(A)$ . Then there exist  $\beta$ -closed subset F lies in A s.t.  $x \in F$ . So,  $x \in X \setminus F$  and  $X \setminus F$  is  $\beta$ -open. Also  $(X \setminus F) \cap A = \phi$ , which contradict our assumption, so  $x \in \beta C(A)$ .

**Theorem 23** If  $A \subset X$ , then

- 1.  $(\eta I(A))^c = \beta C(A^c)$ , where  $\eta I(A)$  means  $\beta$  -interioer of A.
- 2.  $\eta I(A) = (\beta C(A^c))^c$ .

3. 
$$\beta C(A) = (\eta I(A^c))^c$$
.

Proof.

1. Let  $x \in (\eta I(A)))^c$  that means x does not belong to  $\eta I(A)$ . So from this we can conclude that every  $\eta$ -open set U containing x is s.t. U is not lies in A, or we can say that  $U \cap A^c \neq \phi$ . By standard result  $x \in \beta C(A^c)$ . Therefore  $(\eta I(A))^c \subset \beta C(A^c)$ . Now we prove the reverse inequality for that let us take an arbitrary element  $x \in \beta C(A^c)$  then by standard result every  $\beta$ -open set U containing x is s.t.  $U \cap A^c \neq \phi$ , i.e. all  $\beta$  – open set U containing x is s.t. U does not lies in A. Due to  $\eta$ -interior of A, x does not belong to  $\eta I(A)$ , that means  $x \in (\eta I(A))^c$  and  $(\eta I(A))^c \supset \beta (A^c)$  thus  $(\eta I(A))^c = \beta C(A^c)$ .

2. By taking complement on the both sides of the first part we get  $((\eta I(A))^c)^c = (\beta C(A^c))^c$  that is  $\eta I(A) = (\beta C(A^c))^c$ .

3. By replacing  $A^c$  by A in part (1) we get  $(\eta I(A^c))^c = \beta C((A^c)^c)$  that is  $\beta C(A) = (\eta I(A^c))^c$ .

**Remark:** For space X then  $\tau_g^{**} = \{V \subset X : \beta C(X \setminus V) = X \setminus V\}.$ 

**Theorem 24** If GPRWO(X,  $\tau$ ) is a topology, then  $\tau_q^{**}$  is also a topology on the same space

Х.

*Proof.* Clearly  $\phi$  and  $X \in \tau_g^{**}$ . Now let  $\{A_i : i \in \Delta\} \in \tau_g^{**}$  then  $\beta C(X \setminus (\cup A_i)) = gprw - cl(\cap (X - A_i)) \subset \cap \beta C(X \setminus A_i) = \cap (X \setminus A_i) = X \setminus \cup A_i$ . Hence,  $\cup A_i \in \tau_g^{**}$ . Let  $A, B \in \tau_g^{**}$ . Now  $\beta C(X \setminus (A \cap B)) = \beta C((X \setminus A) \cup (X \setminus B)) = \beta C(X \setminus A) \cup \beta C(X \setminus B) = (X \setminus A) \cup (X - B)$ . So  $A \cap B \in \tau_g^{**}$ . Thus  $\tau_g^{**}$  is topology.

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