ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $4(x^2+y^2)-7xy+x+y+1=31z^2$

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ABSTRACT:

The non-homogeneous ternary quadratic Diophantine equation represented by $4(x^2+y^2)-7xy+x+y+1=31z^2$ is studied for finding its non – zero distinct integer solutions.

Keywords: Non -homogeneous, Ternary quadratic equation, Integral solutions.

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INTRODUCTION:

Ternary quadratic equations are rich in variety [1-4, 17-19]. For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, we consider yet another interesting non-homogeneous ternary quadratic equation $4(x^2+y^2)-7xy+x+y+1=31z^2$ and obtain infinitely many non-trivial integral solutions.

METHOD OF ANALYSIS:

Let x, y, z be any three non-zero distinct integers such that

$$4(x^2+y^2)-7xy+x+y+1=31z^2 \tag{1}$$

Introducing the linear transformations

$$x=u+v-1, y=u-v-1 \tag{2}$$

in (1), it leads to

$$u^2 + 15v^2 = 31z^2 \qquad (3)$$

We present below different methods of solving (3) and thus, obtain different patterns of integral solutions to (1).

METHOD-1

(3) is written in the form of ratio as

$$\frac{u+4z}{(z-v)} = \frac{15(z+v)}{u-4z} = \frac{\alpha}{\beta}, \qquad \beta \neq 0 \qquad (4)$$

which is equivalent to the system of equations

$$u\beta + v\alpha + (4\beta - \alpha)z = 0$$

$$15v\beta - u\alpha + (4\alpha + 15\beta)z = 0$$

Solving the above two equations by cross multiplication method, one obtains

$$u = 4\alpha^2 - 60\beta^2 + 30\alpha\beta$$

$$v = \alpha^2 - 15\beta^2 - 8\alpha\beta \qquad (5)$$

$$z = \alpha^2 + 15\beta^2$$

Hence, in view of (2) and (5), the non- zero integral solutions of (1) are given by

$$x = 5\alpha^2 - 75\beta^2 + 22\alpha\beta - 1$$

$$y = 3\alpha^2 - 45\beta^2 + 38\alpha\beta - 1$$

$$z = \alpha^2 + 15\beta^2$$

METHOD-2

In addition to (4), (3) is written in the form of ratio as

$$\frac{u+4z}{15(z-v)} = \frac{(z+v)}{u-4z} = \frac{\alpha}{\beta}, \qquad \beta \neq 0 \qquad (6)$$

which is equivalent to the system of equations

$$u\beta + 15v\alpha + (4\beta - 15\alpha)z = 0$$

$$v\beta - u\alpha + (4\alpha + \beta)z = 0$$

Solving the above two equations by cross multiplication method, one obtains

$$u = 60\alpha^2 - 4\beta^2 + 30\alpha\beta$$

$$v = 15\alpha^2 - \beta^2 - 8\alpha\beta \qquad (7)$$

$$z = 15\alpha^2 + \beta^2$$

Hence, in view of (2) and (7), the non- zero integral solutions of (1) are found to be

$$x = 75\alpha^2 - 5\beta^2 + 22\alpha\beta - 1$$
$$y = 45\alpha^2 - 3\beta^2 + 38\alpha\beta - 1$$
$$z = 15\alpha^2 + \beta^2$$

METHOD-3

Also, (3) is written in the form of ratio as

$$\frac{u+4z}{5(z-v)} = \frac{3(z+v)}{u-4z} = \frac{\alpha}{\beta}, \qquad \beta \neq 0 \qquad (8)$$

which is equivalent to the system of equations

$$u\beta + 5v\alpha + (4\beta - 5\alpha)z = 0$$
$$3v\beta - u\alpha + (4\alpha + 3\beta)z = 0$$

Solving the above two equations by cross multiplication method, one obtains

$$u = 20\alpha^2 - 12\beta^2 + 30\alpha\beta$$
$$v = 5\alpha^2 - 3\beta^2 - 8\alpha\beta \qquad (9)$$
$$z = 5\alpha^2 + 3\beta^2$$

Hence, in view of (2) and (9), the non- zero integral solutions of (1) are

$$\left.\begin{array}{l} x = 25\alpha^2 - 15\beta^2 + 22\alpha\beta - 1 \\ y = 15\alpha^2 - 9\beta^2 + 38\alpha\beta - 1 \end{array}\right\}$$

METHOD-4

Further, (3) is written in the form of ratio as,

$$\frac{u+4z}{3(z-v)} = \frac{5(z+v)}{u-4z} = \frac{\alpha}{\beta}, \qquad \beta \neq 0$$

which is equivalent to the system of equations

$$u\beta + 3v\alpha + (4\beta - 3\alpha)z = 0$$
$$5v\beta - u\alpha + (4\alpha + 5\beta)z = 0$$

Solving the above two equations by cross multiplication method, one obtains

$$u = 12\alpha^2 - 20\beta^2 + 30\alpha\beta$$
$$v = 3\alpha^2 - 5\beta^2 - 8\alpha\beta \qquad (10)$$
$$z = 3\alpha^2 + 5\beta^2$$

Hence, in view of (2) and (10), the non- zero integral solutions of (1) are given by

$$x = 15\alpha^2 - 25\beta^2 + 22\alpha\beta - 1$$
$$y = 9\alpha^2 - 15\beta^2 + 38\alpha\beta - 1$$
$$z = 3\alpha^2 + 5\beta^2$$

METHOD-5

Introduce the linear transformations

$$u = 4W, z = X + 15T, v = X + 31T \quad (11)$$

Substituting (11) in (3), it reduces to

$$X^2 = 465T^2 + W^2 \quad (12)$$

which is satisfied by

$$X = 465r^2 + s^2$$
$$T = 2rs \quad (13)$$
$$W = 465r^2 + s^2$$

Hence, in view of (2), (11) and (13), the non- zero integral solutions of (1) are given by

$$x = 2325r^2 - 3s^2 + 62rs - 1$$
$$y = 1395r^2 - 5s^2 - 62rs - 1$$
$$z = 465r^2 + s^2 + 30rs$$

METHOD-6

Also (3) as $u^2 + 15v^2 = 31z^2$

Write z as $z = \alpha^2 + 15\beta^2$ (14)

Also, 31 is written as $31 = \left(4 + i\sqrt{15}\right)\left(4 - i\sqrt{15}\right)$ (15)

Substituting (14) and (15) in (3) and employing the factorization method, define

$$\left(u + i\sqrt{15}v\right) = \left(4 + i\sqrt{15}\right)\left(\alpha + i\sqrt{15}\beta\right)^2$$

On equating the real and imaginary parts, we have

$$u = 4\alpha^2 - 60\beta^2 - 30\alpha\beta \quad (16)$$
$$v = \alpha^2 - 15\beta^2 + 8\alpha\beta$$

Using (16) in (2) we have

$$\left.\begin{array}{l} x = 5\alpha^2 - 75\beta^2 - 22\alpha\beta - 1 \\ y = 3\alpha^2 - 45\beta^2 - 38\alpha\beta - 1 \end{array}\right\} \quad (17)$$

Thus (17) and (14) represent the non-zero distinct integer solutions to equation (1).

METHOD-7

One may write (3) as

$$u^2 + 15v^2 = 31z^2 * 1 \quad (18)$$

Write 1 as $1 = \frac{(1+i\sqrt{15})(1-i\sqrt{15})}{16}$ (19)

Substituting (15), (19) and (16) in (18) and employing the factorization method, define

$$(u + i\sqrt{15}v) = (4 + i\sqrt{15})(\alpha + i\sqrt{15}\beta)^2 * \frac{(1+i\sqrt{15})}{4}$$

On equating the real and imaginary parts, we have

$$u = \frac{1}{4}(-11\alpha^2 + 165\beta^2 - 150\alpha\beta) \quad (20)$$

$$v = \frac{1}{4}(5\alpha^2 - 75\beta^2 - 22\alpha\beta)$$

Using (20) in (2), we have

$$\left.\begin{aligned} x &= \frac{1}{4}(-6\alpha^2 + 90\beta^2 - 172\alpha\beta - 4) \\ y &= \frac{1}{4}(-16\alpha^2 + 240\beta^2 - 128\alpha\beta - 4) \end{aligned}\right\} \quad (21)$$

As our interest is on finding integer solutions, replacing $\alpha = 2\alpha$ and $\beta = 2\beta$ in (21) and (14), we have

$$x = -6\alpha^2 + 90\beta^2 - 172\alpha\beta - 1$$

$$y = -16\alpha^2 + 240\beta^2 - 128\alpha\beta - 1$$

$$z = 4\alpha^2 + 60\beta^2$$

Thus the above values of x, y and z represent the non-zero distinct integer solutions to equation (1).

NOTE:

It is worth mentioning here that in addition to (19),

1 may be represented as below:

(i) $1 = \frac{(7+i\sqrt{15})(7-i\sqrt{15})}{64}$

(ii) $1=\frac{(7+i4\sqrt{15})(7-i4\sqrt{15})}{289}$

(iii) $1=\frac{(1+i8\sqrt{15})(1-i8\sqrt{15})}{961}$

(iv) $1=\frac{(7+i12\sqrt{15})(7-i12\sqrt{15})}{2209}$

Following the procedure presented as above, for simplicity and brevity, we present below the integer solutions to (1) for (i) to (iv).

Solutions for (i):

$x=12\alpha^2-180\beta^2-152\alpha\beta-1$

$y=\alpha^2-15\beta^2-178\alpha\beta-1$

$z=4\alpha^2+60\beta^2$

Solutions for (ii):

$x=-153\alpha^2+2295\beta^2-12818\alpha\beta-1$

$y=-935\alpha^2+14025\beta^2-10642\alpha\beta-1$

$z=289\alpha^2+4335\beta^2$

Solutions for (iii):

$x=-2573\alpha^2+38595\beta^2-37882\alpha\beta-1$

$y=-4619\alpha^2+69285\beta^2-23498\alpha\beta-1$

$z=961\alpha^2+14415\beta^2$

Solutions for (iv):

$x=-4559\alpha^2+68385\beta^2-91838\alpha\beta-1$

$y=-9729\alpha^2+145935\beta^2-63262\alpha\beta-1$

$z=2209\alpha^2+33135\beta^2$

METHOD-8

Write (12) as the system of double equations as shown in Table 1 below:

Table 1: System of double equations

System	1	2	3	4	5	6	7	8	9	10
X+W	465	T^2	$5T^2$	$15T^2$	$31T^2$	$155T^2$	465T	93T	31T	155T
X-W	T^2	465	93	31	15	3	T	5T	15T	3T

Solving each of the system of equations in Table 1, the corresponding values of X, W and T are obtained. Substituting the values of X, W and T in (11) and (2), the respective values of x, y and z are determined. For simplicity and brevity, the integer solutions to (1) obtained through solving each of the above system of equations are exhibited.

System :1

$x=-6k^2+56k+1191$

$y=-10k^2-72k+663$

$z=2k^2+32k+248$

System:2

$x=10k^2+72k-665$

$y=6k^2-56k-1193$

$z=2k^2+32k+248$

System:3

$x=50k^2+112k-97$

$y=30k^2-32k-257$

$z=10k^2+40k+64$

System :4

System:5

System:6

$x = 150k^2 + 212k + 21$
$y = 90k^2 + 28k - 87$
$z = 30k^2 + 60k + 38$

$x = 310k^2 + 372k + 85$
$y = 186k^2 + 124k - 23$
$z = 62k^2 + 92k + 38$

$x = 1550k^2 + 1612k + 413$
$y = 930k^2 + 868k + 193$
$z = 310k^2 + 340k + 94$

System :7

$x = 1192T - 1$
$y = 664T - 1$
$z = 248T$

System:8

$x = 256T - 1$
$y = 96T - 1$
$z = 64T$

System:9

$x = 86T - 1$
$y = -22T - 1$
$z = 38T$

System :10

$x = 414T - 1$
$y = 194T - 1$
$Z = 94T$

REMARKABLE OBSERVATION :I

- If the non-zero integer triplet (u_0, v_0, z_0) is any solution of (3) then each of the following three triplets of integer based on u_0, v_0 and z_0 also satisfies (1).

Triplet: 1 (x_n, y_n, z_n)

$$x_n = 16^n u_0 + \left[[15(16)^{n-1} + 31(-16)^{n-1}]v_0 - 31[(16)^{n-1} + (-16)^{n-1}]z_0\right] - 1$$
$$y_n = 16^n u_0 + \left[[15(16)^{n-1} + 31(-16)^{n-1}]v_0 - 31[(16)^{n-1} + (-16)^{n-1}]z_0\right] - 1,$$
$$z_n = 15[(16)^{n-1} + (-16)^{n-1}]v_0 - [31(16)^{n-1} + 15(-16)^{n-1}]z_0$$

Triplet: 2 (x_n, y_n, z_n)

$$x_n = \left[[25(3)^n + 31(-3)^n]u_0 - 155[(3)^n - (-3)^n]z_0\right] + 3^n v_0 - 1$$
$$y_n = \left[[25(3)^n + 31(-3)^n]u_0 - 155[(3)^n - (-3)^n]z_0\right] - 3^n v_0 - 1$$
$$z_n = 5[(3)^n - (-3)^n]u_0 + [-31(3)^n + 25(-3)^n]z_0$$

Triplet: 3 (x_n, y_n, z_n)

$$x_n = \frac{1}{43}\left[\left[(-43)(-29)^n u_0 - 30[(14)^n - (-29)^n]v_0\right] + \left[(-4)[(14)^n - (-29)^n]u_0 - 43(14)^n v_0\right]\right] - 1$$
$$y_n = \frac{1}{43}\left[\left[(-43)(-29)^n u_0 - 30[(14)^n - (-29)^n]v_0\right] + \left[(-4)[(14)^n - (-29)^n]u_0 - 43(14)^n v_0\right]\right] - 1$$
$$z_n = 16^n z_0$$

REMARKABLE OBSERVATION : II

- Consider x and y to be the length and breadth of a Rectangle R, whose Area, Perimeter and Length of the diagonal are represented by A, P and L respectively.

Then it is noted that

1. $8L^2 - 14A + P \equiv -2 \pmod{62}$
2. $31[8L^2 - 14A + P + 2]$ is a perfect square.
3. $93\{8L^2 - 14A + P + 2\}$ is a Nasty number.

CONCLUSION:

To conclude, one may search for other patterns of solutions and their corresponding properties.

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