BERNSTEIN TYPE INEQUALITIES FOR POLAR DERIVATIVE OF POLYNOMIAL

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Abstract- If $p(z)$ ***is a polynomial of degree n such that*** $p(z)\neq 0$ ***in*** $|z|<k$, $k\leq 1$, ***then Govil [Proc. Nat. Acad. Sci., Vol. 50, pp. 50-52, 1980.] proved***

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|,$$

provided $|p'(z)|$ ***and*** $|q'(z)|$ ***attain their maxima at the same point on the circle*** $|z|=1$, ***where*** $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Equality in the above inequality holds for $p(z)=z^n+k^n$.

In this paper, we extend the above inequality and an improved version of this into polar derivative of a polynomial.

Keywords – Polynomial, Polar Derivative of a polynomial, Inequalities, Maximum Modulus.

I. Introduction

It was for the first time, Bernstein [10, 11] investigated an upper bound for the maximum modulus of the first derivative of a complex polynomial on the unit circle in terms the maximum modulus of the polynomial on the same circle and proved the following famous result known as Bernstein's inequality that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)|\leq n\max_{|z|=1}|p(z)|. \tag{1.1}$$

Inequality (1.1) is best possible and equality occurs for $p(z)=\lambda z^n$, $\lambda\neq 0$, is any complex number.

If we restrict to the class of polynomials having no zero in $|z|<1$, then inequality (1.1) can be sharpened as

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{2}\max_{|z|=1}|p(z)| . \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.

Inequality (1.2) was conjectured by Erdös and later proved by Lax [8].

Simple proofs of this theorem were later given by de-Bruijn [5], and Aziz and Mohammad [2].

It was asked by R.P. Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z|<k$, $k>0$, then how large can

$$\left\{ \max_{|z|=1}|p'(z)| \Big/ \max_{|z|=1}|p(z)| \right\} \text{ be ?} \tag{1.3}$$

A partial answer to this problem was given by Malik [9], who proved

Theorem A. *If $p(z)$ is a polynomial of degree n having no zero in the disc $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.4}$$

The result is best possible and equality holds for $p(z)=(z+k)^n$.

For the class of polynomials not vanishing in $|z|<k, k\leq 1$, the precise estimate for maximum of $|p'(z)|$ on $|z|=1$, in general, does not seem to be easily obtainable.

For quite some time, it was believed that if $p(z)\neq 0$ in $|z|<k$, $k\leq 1$, then the inequality analogous to (1.4) should be

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{1.5}$$

till E.B. Saff gave the example $p(z)=\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)$ to counter this belief.

Govil [6] obtained inequality (1.5) with extra condition. More precisely, he proved the following

Theorem B. *If $p(z)$ is a polynomial of degree n such that $p(z)\neq 0$ in $|z|<k$, $k\leq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|, \tag{1.6}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z|=1$, where $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Equality in (1.6) *holds for $p(z)=z^n+k^n$.*

Aziz and Rather [3] further improved the bound of (1.6) by involving $\min_{|z|=k}|p(z)|$.

Theorem C. *If $p(z)$ is a polynomial of degree n such that $p(z)\neq 0$ in $|z|<k$, $k\leq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right\}, \tag{1.7}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z|=1$, where $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

As in Theorem B, equality in (1.6) *occurs for $p(z)=z^n+k^n$.*

Let $p(z)$ be a polynomial of degree n and α be any real or complex number, the polar derivative of $p(z)$, denoted by $D_\alpha p(z)$, is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{1.8}$$

The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ of $p(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z). \tag{1.9}$$

It is of interest to extend ordinary derivative inequalities into polar derivative of a polynomial, for the later version is a generalization of the first. In this direction, Aziz and Shah [4] for the first time extended (1.1) to polar derivative by proving

Theorem D. *If $p(z)$ is a polynomial of degree n then for every real or complex number α with $|\alpha| \ge 1$,*

$$\max_{|z|=1} \left|D_\alpha p(z)\right| \le n|\alpha| \max_{|z|=1} \left|p(z)\right|. \tag{1.10}$$

Inequality (1.10) *becomes equality for $p(z) = a z^n$, $a \ne 0$.*

Further, Aziz [1] extended inequality (1.2) to polar derivative.

Theorem E. *If $p(z)$ is a polynomial of degree n having no zero in the disc $|z| < 1$, then for every real or complex number α with $|\alpha| \ge 1$,*

$$\max_{|z|=1} \left|D_\alpha p(z)\right| \le \frac{n}{2}\left(|\alpha| + 1\right) \max_{|z|=1} \left|p(z)\right|. \tag{1.11}$$

The result is best possible and extremal polynomial is $p(z) = z^n + 1$.

For the class of polynomials not vanishing in the disc $|z| < k$, $k \ge 1$, Aziz [1] obtained the extension of Theorem A to polar derivative of $p(z)$.

Theorem F. *If $p(z)$ is a polynomial of degree n having no zero in the disc $|z| < k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge 1$,*

$$\max_{|z|=1} \left|D_\alpha p(z)\right| \le n\left(\frac{k + |\alpha|}{1 + k}\right) \max_{|z|=1} \left|p(z)\right|. \tag{1.12}$$

The result is best possible and equality in (1.12) *holds for the polynomial* $p(z)=(z+k)^n$, *with* $\alpha \geq 1$.

In this paper, we extend both the Theorems B and C into polar derivative of a polynomial. More precisely, we prove

Theorem 1. *If* $p(z)$ *is a polynomial of degree n such that* $p(z)\neq 0$ *in* $|z|<k$, $k\leq 1$, *then for every real or complex number* α *with* $|\alpha|\geq 1$,

$$\max_{|z|=1}\left|D_{\alpha}p(z)\right|\leq n\frac{\left(k^n+|\alpha|\right)}{1+k^n}\max_{|z|=1}\left|p(z)\right|, \tag{1.13}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on the circle* $|z|=1$, *where* $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Dividing both sides of (1.13) by $|\alpha|$ and making limit as $|\alpha|\to\infty$, we obtain inequality (1.6).

Next, we prove the polar derivative form of Theorem C.

Theorem 2. *If* $p(z)$ *is a polynomial of degree n such that* $p(z)\neq 0$ *in* $|z|<k$, $k\leq 1$, *then for every real or complex number* α *with* $|\alpha|\geq 1$,

$$\max_{|z|=1}\left|D_{\alpha}p(z)\right|\leq \frac{n}{1+k^n}\left\{\left(k^n+|\alpha|\right)\max_{|z|=1}\left|p(z)\right|-\left(|\alpha|-1\right)\min_{|z|=k}\left|p(z)\right|\right\}, \tag{1.14}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on the circle* $|z|=1$, *where* $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Dividing both sides of (1.14) by $|\alpha|$ and making limit as $|\alpha|\to\infty$, we get inequality (1.7).

II LEMMA

The following lemma is needed for the proofs of the theorems.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree n, then on* $|z|=1$,

$$|p'(z)|+|q'(z)|\leq n\max_{|z|=1}|p(z)|, \tag{2.1}$$

where

$$q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}.$$

The above lemma is a special case of a result due to Govil and Rahman [7].

III. PROOF OF THE THEOREM

Proof of Theorem 1. We omit the proof as it follows on the same lines as that of Theorem 2 by using Theorem B, instead of Theorem C.

Proof of Theorem 2. Let $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. Then it is easy to verify that for $|z| = 1$,

$$|q'(z)| = |np(z) - zp'(z)| \; . \tag{3.1}$$

Now, for every real or complex number α, the polar derivative of $p(z)$ with respect to α is

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

This implies for $|z| = 1$,

$$\begin{aligned} |D_\alpha p(z)| &\le |np(z) - zp'(z)| + |\alpha||p'(z)| \\ &= |q'(z)| + |\alpha||p'(z)| \quad \text{(by (3.1))} \\ &= |q'(z)| + |p'(z)| - |p'(z)| + |\alpha||p'(z)|. \\ &\le n\max_{|z|=1}|p(z)| + (|\alpha| - 1)|p'(z)| \quad \text{(Lemma 2.1)} \end{aligned} \tag{3.2}$$

Since $|\alpha| \ge 1$, the above inequality when combined with inequality (1.7) of Theorem C gives

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| &\le n\max_{|z|=1}|p(z)| + (|\alpha| - 1)\left\{\frac{n}{1+k^n}\left(\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right)\right\} \\ &= \frac{n}{1+k^n}\left\{(k^n + |\alpha|)\max_{|z|=1}|p(z)| - (|\alpha| - 1)\min_{|z|=k}|p(z)|\right\} \; . \end{aligned}$$

This completes the proof of Theorem 2.

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