On 2-Odd Labeling Of Graphs

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***Abstract:* A graph G is a 2-odd graph if its vertices can be labeled with distinct integers such that for any two adjacent vertices, the absolute difference of their labels is either an odd integer or 2. In this paper, we investigate 2-odd labeling of various classes of graphs.**

Keywords: Distance Graphs, 2-odd Graphs

1. Introduction

All the graphs considered in this paper are finite, simple, connected, and undirected. By $G(V,E)$, or simply G, we mean a graph G with vertex set V and edge set E. According to Laison et al. [3] a graph G is 2-odd if there exists a one-to-one labeling $h: V(G) \to Z$ ("the set of all integers") such that for any two adjacent vertices u and v, the integer $|h(u) - h(v)|$ is either an odd integer or exactly 2. They also defined that $h(uv) = |h(u) - h(v)|$ and called h a 2-odd labeling of G. So G is a 2-odd graph if and only if there exists a 2-odd labeling of G. Note that in a 2-odd labeling, the vertex labels of G must be distinct, but the edge labels need not be so. Furthermore, by this definition $h(uv)$ may still be either 2 or odd if uv is not an edge of G. In this paper, we obtain 2-odd labeling of some graphs.

Note 1:
A 2-odd labeling of a graph G is not unique.

Example 1:
A path P_n can be labeled with $5, 10, \ldots, 5n$ or $2, 4, \ldots, 2n$.

2. Main Results

In this section, we establish the 2-odd labeling of certain graphs. We also make use of some well- known number theoretic concepts in proving some of the results.

Lemma 1. [3]
If a graph G has a subgraph that does not admit a 2-odd labeling then G cannot have a 2-odd labeling.

One can notice that the Lemma 1 clearly holds since if G has a 2-odd labeling then this will yield a 2-odd labeling for any subgraph of G.

Theorem 1. [3]
1. Every bipartite graph is 2-odd.
2. Every cycle is 2-odd.

Definition 1. [6]
The wheel graph $W_n = C_{n-1} \wedge K_1$ is a graph with n vertices ($n \geq 4$), formed by joining the central vertex K_1 to all the vertices of C_{n-1}.

Definition 2. [5]
A helm graph, denoted H_n is a graph obtained by attaching a single edge and vertex to each vertex of the outer circuit of a wheel graph W_n.
Theorem 2.
The helm graph H_n admits a 2-odd labeling for all $n \geq 4$.
Proof.
Let W_n be the given wheel graph on $n \geq 4$ vertices which consists of $n-1$ vertices on the rim, namely $v_1, v_2, \ldots, v_n$ and a central vertex, say v_0. Obtain the helm graph H_n by joining a pendent edge at each vertex of the cycle. We denote the corresponding newly added vertices as $u_1, u_2, \ldots, u_n$. Now define a one-to-one function $f: V(H_n) \to Z$ as follows: without loss of generality, let $f(v_0) = 1$, $f(v_1) = 3$, $f(v_2) = 4$, and $f(v_i) = f(v_{i-1}) + 2$ for $3 \leq i \leq n$. Let k be the first even number sufficiently larger number than $f(v_n)$. Then $f(u_1) = k$, $f(u_2) = k + 1$, and $f(u_i) = f(u_{i-1}) + 2$ for $3 \leq i \leq n$. An easy check shows that f is the required 2-odd labeling of H_n as $|f(v_0) - f(v_i)|$ are all odd numbers for $3 \leq i \leq n$, $|f(v_0) - f(v_1)| = 2$, $|f(v_0) - f(v_2)| = 3$, $|f(v_i) - f(v_{i+1})| = 2$ for $2 \leq i \leq n-1$, $|f(v_n) - f(v_1)|$ is an odd integer, and $|f(v_i) - f(u_i)|$ are all odd integers, for $1 \leq i \leq n$.
Corollary 1.
The wheel graph W_n admits a 2-odd labeling for all $n \geq 4$.

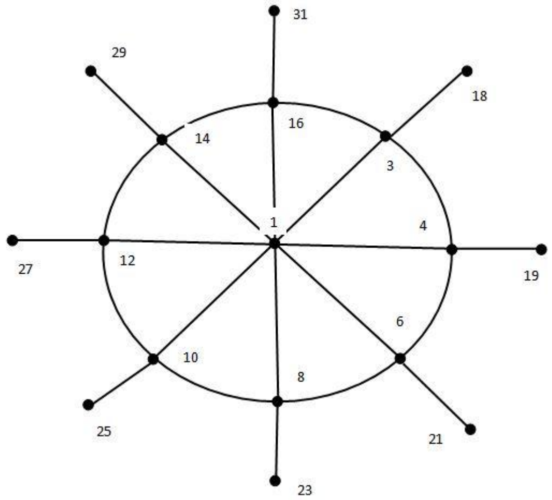

Figure 1. A 2-odd labeling of helm graph H_9

Definition 3. [9]
A double-wheel graph DW_n of size n is composed of $2C_n \wedge K_1$. In other words, DW_n consists of two cycles, each of size n, where the vertices of the two cycles are all connected to a common hub represented by K_1.
Theorem 3.
The double wheel graph DW_n admits a 2-odd labeling for $n \geq 3$.
Proof.
Let DW_n be the given double wheel graph with $n \geq 3$. We label the central vertex as v_0, the inner cycle vertices as $v_1, v_2, \ldots, v_n$, and the outer cycle vertices as $u_1, u_2, \ldots, u_n$. Define a one-to-one labeling $h: V(DW_n) \to Z$ as follows: without loss of generality, let $h(v_0) = 1$, $h(v_1) = 3$, $h(v_2) = 4$, and $h(v_i) = h(v_{i-1}) + 2$ for $3 \leq i \leq n$. Also define $h(u_i) = -h(v_i)$ for $1 \leq i \leq n$. One can clearly observe that h induces the required 2-odd labeling of DW_n for all $n \geq 3$.

Definition 4. [1]
An umbrella graph $U(m,n)$ is defined as a graph with $V\left(U(m,n)\right) = \{v_1, v_2, \dots, v_{m+n}\}$ and $E\left(U(m,n)\right) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq n-1\}, E_2 = \left\{v_n v_{n+j} : 1 \leq j \leq m\right\}$, and $E_3 = \{v_k v_{k+1} : n+1 \leq k \leq m+n-1\}$.
Theorem 4.
An umbrella graph $U(m,n)$ admits a 2-odd labeling for any $m, n \in N$.
Proof.
Let $U(m,n)$ be the given umbrella graph on $m, n \geq 1$ vertices. We label the vertex common vertex as v_0, vertices of a path P_m as $v_1, v_2, \dots, v_m$, and vertices of the handle as $u_1, u_2, \dots, u_n$. One can note that $v_0 = u_1$ and $|U(m,n)| = m+n+1$. Now define an injective labeling $h: V(U(m,n)) \to Z$ as follows: without loss of generality, let $h(v_0) = 1$ and $h(v_i) = 2i$ for $1 \leq i \leq m$. Next let $2k$ be any even number sufficiently greater than $h(v_m)$. Then define $h(u_2) = 2k$ and $h(u_i) = 2h(u_{i-1})$ for $3 \leq i \leq n$. An easy check shows that h is the desired 2-odd labeling of $U(m,n)$ for any $m, n \in N$.
Definition 5: [7]
Duplication of a vertex v_k of a graph H by an edge $e = v'_k v''_k$ is obtained by adding two new vertices v'_k, v''_k and an edge $e = v'_k v''_k$ such that $N({v_k}') = \{v_k, {v_k}''\}$ and $N({v_k}'') = \{v_k, {v_k}'\}$. The resultant graph is denoted by G.
Conjecture 1: [2] (**The Twin Prime Conjecture**)
There are infinitely many pairs of primes that differ by 2.
Theorem 5:
The graph obtained by performing duplication of a vertex by an edge at all the vertices of a 2-odd graph admits a 2-odd labeling if the Twin prime conjecture is true.
Proof.
Let H be the given 2-odd graph with a 2-odd labeling h and $V(H) = \{v_1, v_2, \dots, v_n\}$. Let $h(v_j) = \max_{1 \leq i \leq n} h(v_i)$. Obtain the graph G by performing duplication of every vertex v_k by an edge $v'_k v''_k$ for $k = 1, 2, \dots, n$. One can see that G is a graph with $3n$ vertices and having n vertex disjoint cycles of length 3. Define a one-to-one labeling $g : V(G) \to Z$ as follows: let $g(v_i) = h(v_i)$ for all $1 \leq i \leq n$. Let p'_1 and p''_1 be any twin primes sufficiently larger than $h(v_j)$. Let $g(v'_j) = h(v_j) + p'_1$ and $g\left(v''_j\right) = h\left(v_j\right) + p''_1$. Next let p'_2, p''_2 be any twin primes sufficiently larger than $g\left(v''_j\right)$. Now let $h(v_l) = \max_{1 \leq i \leq n, i \neq j} h(v_i)$. Let $g(v'_l) = h(v_l) + p'_2$ and $g(v''_l) = h(v_l) + p''_2$. Continuing the same process for all the vertices of G, one can obtain 2-odd labeling of G as there are infinitely many pairs of primes that differ by 2 by the Twin prime conjecture.
Definition 6. [8]
The join of two graphs H_1 and H_2, denoted by $H_1 + H_2$, is a graph formed with $V(H_1 + H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 + H_2) = E(H_1) \cup E(H_2) \cup \{uv : u \in V(H_1), v \in V(H_2)\}$.
Definition 7. [10-12]
The complete graph K_n is a graph in which any two vertices are adjacent.
Theorem 6.
The graph $K_2 + mK_1$ admits a 2-odd labeling for all $m \in N$.
Proof.
Let K_1 and K_2 be the given complete graphs on 1 and 2 vertices, respectively. We label the vertices of K_2 as v_1 and v_2. We also label the vertices of mK_1 as $u_1, u_2, \dots, u_m$. Obtain $K_2 + mK_1$ by using Definition 6. Now define a one-one function $h: V(K_2 + mK_1) \to Z$ as follows: without loss of generality, let $h(v_1) = 1$ and $h(v_2) = -1$. Also define $h(u_i) =$

$2i$ for $1 \leq i \leq m$. One can see that h is the required 2-odd labeling of $K_2 + mK_1$ for any $m \in N$.

Conjecture 2. [2] (**Goldbach's Conjecture**)

Any even number $2n$ for $n \geq 2$ is the sum of two primes.

Definition 8. [4]

A graph $B_{n,m}$ $(n, m \in N)$ is said to be a butterfly graph if two cycles of same order $n \geq 3$ sharing a common vertex with an arbitrary number m of pendant edges attached at the common vertex.

Theorem 7.

A butterfly graph $B_{n,m}$ permits a 2-odd labeling for any $n \geq 3, m \in N$ if Goldbach's conjecture is true.

Proof.

Let $B_{n,m}$ be the given butterfly graph with $n \geq 3$ and $m \in N$. We call the vertex set of $B_{n,m}$ as $V\left(B_{n,m}\right) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{w_1, w_2, \ldots, w_m\}$. Clearly $\left|V\left(B_{n,m}\right)\right| = 2n - 1 + m$ as $u_1 = v_1$. We consider the following two cases.

Case 1: C_n, when n is even

The proof is direct from Theorem 1.

Case 2: C_n, when n is odd

Define an injective labeling $h: V\,(B_{n,m}) \rightarrow Z$ as follows: without loss of generality, let $h(v_i) = 2(i-1)$ for $1 \leq i \leq n-1$. Next by Goldbach's conjecture, $h(v_{n-1}) = p_1 + p_2$ and let $h(v_n) = p_1$ or p_2. Also define $h(u_i) = -h(v_i)$ for all $2 \leq i \leq n$. Finally, let $l_1, l_2, \ldots, l_m$ be sufficiently larger odd numbers than $h(v_{n-1})$ and $h(w_i) = l_i$ for all $1 \leq i \leq m$. A simple check shows that the vertex labels are distinct and h induces a 2-odd labeling of $B_{n,m}$.

Definition 9. [7]

Duplication of a vertex v of a graph H by a vertex is obtained by adding a new vertex w to H and adding edges in such a way that $N(w) = N(v)$. The resultant graph is denoted by G.

Theorem 8.

The graph obtained from a cycle C_n by performing duplication of a vertex by a vertex at all the vertices of C_n admits a 2-odd labeling if Goldbach's conjecture is true.

Proof.

Let $C_n: v_1 v_2, \ldots, v_n v_1$ be the given cycle. Obtain the graph G by performing duplication of a vertex by a vertex at all the vertices of C_n with $V\,(G) = V\,(C_n) \cup \{u_i: 1 \leq i \leq n\}$. Define an injective function $h: V\,(G) \rightarrow Z$ as follows: $h(v_i) = 2(i-1)$ for $1 \leq i \leq n-1$. If Goldbach's conjecture is true then $h(v_{n-1})$ can be expressed as a sum of two primes. Let $h(v_{n-1}) = p_1 + p_2$ and without loss of generality, let $h(v_n) = p_1 > 2$ and $h(u_n) = p_2$. Next let $h(u_1) = h(u_1) - 2$, $h(u_1) = h(u_{n-1}) + 2$, and $h(u_i) = 2m + 1$ for $2 \leq i \leq n - 2$ and any sufficiently large $m \in N$. One can easily see that h is the desired 2-odd labeling of G.

3. Conclusion

The 2-odd labeling of various classes of graphs such as helm graph, umbrella graph are established. Investigating 2-odd labeling of other classes of graphs is still open and this is for future research. One can also explore the exclusive applications of 2-odd labeling in real life situations.

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