On Sum Of Positive Integral Powers Of Natural Numbers

B. Mahaboob1, V.B.V.N. Prasad2,G. Balaji Prakash3,B.V.Appa Rao4, M.Rajaiah5, C.Narayana6

1,2,3,4Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur Dist., A.P. India.
5Audisankara College of Engineering & Technology, Gudur, A.P., India. 6Sri Harsha Institute of P. G. Studies, Nellore, SPSR Nellore, A.P., India.
6Sri Harsha Institute of P. G. Studies, Nellore, SPSR Nellore, A.P., India.

Abstract:
The main objective of this paper is to show the sum of positive integral powers say kth powers of first n-natural numbers coincides with a polynomial function of degree k+1 in n over natural numbers for any positive integer k. This has been an interesting problem to the mathematicians for decades. In this research article, the existence and uniqueness of the polynomial has been depicted with principles of Linear Algebra. Some astonishing results among the polynomial coefficients have been traced. Moreover the methods depicted here open a way to write a formula for sum of any positive integral powers of first n-natural numbers.

Keywords: Rank of a matrix, Coefficient matrix, Augmented matrix, Variable matrix, Non-homogeneous linear equations.

1. INTRODUCTION

Thomas Harrlot (1560-1621) was the first mathematician who gave the generalized form of sum of positive integral powers of first n- natural numbers. Johann Faulhaber (1580-1635), a German mathematician, proposed formulas up to 17th power and his work was considered a master piece at that time. However Johann Faulhaber failed to generalize his results. Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1705) were credited with the innovation of these results in explicit form. But Jacob Bernoulli (1654-1705) gave the most significant generalized formula explicitly.

In 2012, Dohyoung Ryang and Tony Thompson, in their research article, generated a formula for sum of positive integral powers of first n- natural numbers.

Janet Beery, in 2010, in his paper, discussed the sum of positive integral powers of first n – natural numbers.

Do Tan Si,in 2019,in his research article proposed tables to compute Bernoulli numbers which are used in the generalization of sum of positive integral powers of first n- natural numbers

First consider the identity $\mathbf{(r+1)^2 = r^2 + 2r + 1}$

Let r be a natural number

$(r+1)^2 - r^2 = 2r + 1$

Taking $\sum_{r=1}^{n}$ on both sides

$$\sum_{r=1}^{n}(r+1)^2 - r^2 = \sum_{r=1}^{n} 2r + 1$$

$$(2^2 - 1^2) + (3^2 - 2^2) + \cdots + ((n+1)^2 - n^2) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$(n+1)^2 - 1^2 = 2\sum n + n$$

$$n^2 + 2n + 1 - 1 = 2\sum n + n$$
$$n^2 + n = 2\sum n$$
$$\sum n = \frac{n^2 + n}{2}$$
$$\sum n = \frac{1}{2}n^2 + \frac{1}{2}n \ldots\ldots\ldots\ldots\ldots (1)$$
$$\sum n = \frac{n(n+1)}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
Now we take
$$(r+1)^3 = r^3 + 3r^2 + 3r + 1$$
$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$
Taking $\sum_{r=1}^{n}$ on both sides
$$\sum_{r=1}^{n}((r+1)^3 - r^3) = \sum_{r=1}^{n}(3r^2 + 3r + 1)$$
$$(2^3 - 1^3) + (3^3 - 2^3) + \cdots + ((n+1)^3 - n^3) = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$
$$(n+1)^3 - 1^3 = 3\sum n^2 + 3\sum n + n$$
$$n^3 + 3n^2 + 3n + 1 - 1 = 3\sum n^2 + 3\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) + n$$
$$n^3 + 3n^2 + 3n = 3\sum n^2 + \frac{3}{2}n^2 + \frac{5}{2}n$$
$$3\sum n^2 = n^3 + 3n^2 + 3n - \frac{3}{2}n^2 - \frac{5}{2}n$$
$$3\sum n^2 = n^3 + \frac{3n^2}{2} + \frac{n}{2}$$
$$\sum n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \ldots\ldots\ldots\ldots.. (3)$$
$$\sum n^2 = \frac{2n^3 + 3n^2 + n}{6}$$
$$\sum n^2 = \frac{n(2n^2 + 3n + 1)}{6}$$
$$\sum n^2 = \frac{n(n+1)(2n+1)}{6} \ldots\ldots\ldots\ldots\ldots.. (4)$$
As $(r+1)^4 = r^4 + 4r^3 + 6r^2 + 4r + 1$
$$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$$
Taking $\sum_{r=1}^{n}$ on both sides
$$\sum_{r=1}^{n}(r+1)^4 - r^4 = \sum_{r=1}^{n}(4r^3 + 6r^2 + 4r + 1)$$
$$(2^4 - 1^4) + (3^4 - 2^4) + \cdots + ((n+1)^4 - n^4) = 4\sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$
$$(n+1)^4 - 1^4 = 4\sum n^3 + 6\sum n^2 + 4\sum n + n$$
$$n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 = 4\sum n^3 + 6\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right] + 4\left[\frac{n^2}{2} + \frac{n}{2}\right] + n$$
$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum n^3 + 2n^3 + 3n^2 + n + 2n^2 + 2n + n$$
$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum n^3 + 2n^3 + 5n^2 + 4n$$

$$4\sum n^3 = n^4 + 4n^3 + 6n^2 + 4n - 2n^3 - 5n^2 - 4n$$
$$4\sum n^3 = n^4 + 2n^3 + n^2$$
$$\sum n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \ldots\ldots\ldots\ldots\ldots\ldots(5)$$
$$\sum n^3 = \frac{n^4 + 2n^3 + n^2}{4}$$
$$\sum n^3 = \frac{n^2(n^2 + 2n + 1)}{4}$$
$$\sum n^3 = \frac{n^2(n+1)^2}{4} \ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

From the methods discussed above one can go on computing the formulas for $\sum n^4, \sum n^5, \ldots$

2. RECURRENCE RELATION

Let k be a natural number

$$(r+1)^k = k_{c_0} r^k + k_{c_1} r^{k-1} + k_{c_2} r^{k-2} + \cdots + k_{c_{k-1}} r + k_{c_k}$$
$$(r+1)^k - r^k = k_{c_1} r^{k-1} + k_{c_2} r^{k-2} + \cdots + k_{c_{k-1}} r + k_{c_k}$$

Taking $\sum_{r=1}^{n}$ on both sides

$$\left(2^k - 1^k\right) + \left(3^k - 2^k\right) + \cdots \left((n+1)^k - n^k\right) = k_{c_1}\sum r^{k-1} + k_{c_2}\sum r^{k-2} + \cdots + k_{c_{k-1}}\sum r + k_{c_k}\sum 1$$
$$(n+1)^k - 1^k = k_{c_1}\sum r^{k-1} + k_{c_2}\sum r^{k-2} + \cdots + k_{c_{k-1}}\sum r + k_{c_k}\sum 1$$
$$k_{c_1}\sum r^{k-1} = (n+1)^k - 1 - k_{c_2}\sum r^{k-2} - \cdots - k_{c_{k-1}}\sum r - k_{c_k}\sum 1$$

Replacing k by k+1

$$(k+1)\sum r^k = (n+1)^{k+1} - 1 - (k+1)_{c_2}\sum r^{k-1} - \cdots - (k+1)_{c_k}\sum r - (k+1)_{c_{(k+1)}}\sum 1$$
$$(k+1)\sum r^k = (n+1)^{k+1} - 1 - (k+1)_{c_0}\sum r^0 - (k+1)_{c_1}\sum r \ldots - (k+1)_{c_{(k-1)}}\sum r^{k-1}$$
$$\sum r^k = \frac{1}{(k+1)}\left((n+1)^{k+1} - 1 - (k+1)_{c_0}\sum r^0 - (k+1)_{c_1}\sum r \ldots - (k+1)_{c_{(k-1)}}\sum r^{k-1}\right)$$
(7)

This recurrence relation give the formulas to compute $\sum n, \sum n^2, \sum n^3, \sum n^4, \sum n^5 \ldots$

For instance, if k=1

$$\sum r^1 = \sum r = \frac{1}{2}\left[(n+1)^2 - 1 - 2c_0\sum r^0\right]$$
$$= \frac{1}{2}[n^2 + 2n + 1 - 1] = \frac{1}{2}(n^2 + n) = \frac{n(n+1)}{2}$$

If k=2, then

$$\sum r^2 = \frac{1}{3}\left[(n+1)^3 - 1 - 3c_0\sum r^0 - 3c_1\sum r\right]$$
$$= \frac{1}{3}\left[n^3 + 3n^2 + 3n + 1 - 1 - n - \frac{3n(n+1)}{2}\right]$$
$$= \frac{1}{3}\left[n^3 + 3n^2 + 2n - \frac{3n^2}{2} - \frac{3n}{2}\right]$$
$$= \frac{1}{3}\left(n^3 + \frac{3n^2}{2} + \frac{n}{2}\right)$$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$
$$= \frac{2n^3 + 3n^2 + n}{6}$$
$$= \frac{n(2n^2 + 3n + 1)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6}$$

3. **COINCIDENCE WITH POLYNOMIALS**

It is very interesting to note that sum of positive integral powers say kth powers of first n-natural numbers equals a polynomial of degree (k+1)over natural numbers. This can be seen by the principal of Compute Mathematical Induction from (7)
Moreover (7) gives

$$\sum_{r=1}^{n} r^k =$$
$$= \frac{1}{k+1}\Big[(k+1)_{c_0} n^{k+1} + (k+1)_{c_1} n^k + (k+1)_{c_2} n^{k-1} + \cdots + (k+1)_{c_k} n + (k+1)_{c_{(k+1)}} - 1 - (k+1)_{c_0} \sum r^0 - (k+1)_{c_1} \sum r - \cdots - (k+1)c_{(k-1)} \sum r^{k-1}\Big]$$
$$= \frac{1}{k+1}\Big[n^{k+1} + (k+1)_{c_1} n^k + (k+1)_{c_2} n^{k-1} + \cdots + (k+1)c_k n + \text{a polynomial of degree k in n}\Big]$$

This tells us some interesting properties, one of them is "sum of kth power of first n-natural numbers is a polynomial of degree k+1 in n with no constant term and the leading coefficient is $\frac{1}{k+1}$"
Hence one can assume

$$\sum_{r=1}^{n} r^k = A_1 n + A_2 n^2 + \cdots + A_{k+1} n^{k+1}$$

For n=1, we get $1 = A_1 + A_2 + \cdots + A_{k+1}$
This provides the second property "sum of all coefficients in the polynomial is 1"

4. **IDENTIFYING THE PATTERN**:

Consider the following triangular arrangements

1
1/2 1/2
1/3 1/2 1/6
1/4 1/2 1/4 0
1/5 1/2 1/3 0 -1/30
1/6 1/2 5/12 0 -1/12 0
- - - - - - - - - - -
- - - - - - - - - - -

The first row contains 1 which is the leading coefficient of the polynomial of

$$\sum n^0 = 1(n^{0+1}) = n$$

Now the leading coefficients of $\sum n, \sum n^2, \sum n^3, \sum n^4, \ldots \ldots \ldots$ are obtained by multiplying

1 by $\frac{1}{1+1}=\frac{1}{2}$, $\frac{1}{2}$ by $\frac{2}{2+1}=\frac{1}{3}$, $\frac{1}{3}$ by $\frac{3}{3+1}=\frac{1}{4}$, $\frac{1}{4}$ by $\frac{4}{4+1}=\frac{1}{5}$ etc.

In general the leading coefficient of $\sum n^r$ is got by multiplying the leading coefficient of $\sum n^{r-1}$ by $\frac{r}{r+1}$

Assume $\sum n^0 = a_0 n$

$\sum n^1 = a_0 n^2 + a_1 n$

$\sum n^2 = a_0 n^3 + a_1 n^2 + a_2 n$

$\sum n^3 = a_0 n^4 + a_1 n^3 + a_2 n^2 + a_3 n$

$\sum n^4 = a_0 n^5 + a_1 n^4 + a_2 n^3 + a_3 n^2 + a_4 n$

… … … … … … … …

… … … … …. … … …

Now $\sum n^1 = a_0 n^2 + a_1 n = \frac{1}{2} n^2 + a_1 n$

As sum of coefficients is 1, one can see $a_1 = \frac{1}{2}$

Therefore $\sum n^1 = \frac{1}{2} n^2 + \frac{1}{2} n$

As said above the calculation of all a_0's are clear

Take first a_1 i.e. $\frac{1}{2}$ (which is of $\sum n^1$)

The next a_1's are calculated by multiplying $\frac{1}{2}$ by $\frac{2}{2}=\frac{1}{2}$, $\frac{1}{2}$ by $\frac{3}{3}=\frac{1}{2}$, $\frac{1}{2}$ by $\frac{4}{4}=\frac{1}{2}$, $\frac{1}{2}$ by $\frac{5}{5}=\frac{1}{2}$ etc.

So all a_1's are $\frac{1}{2}$

Now we find first a_2 (of $\sum n^2$)

$$\sum n^2 = a_0 n^3 + a_1 n^2 + a_2 n$$

$$\sum n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + a_2 n$$

$$\frac{1}{3} + \frac{1}{2} + a_2 n = 1 \rightarrow a_2 = \frac{1}{6}$$

So $\sum n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$

$$a_1 \text{ of } \sum n^2 \text{ is } \left(a_1 \text{of} \sum n\right) * \frac{2}{2} = \frac{1}{2} * \frac{2}{2} = \frac{1}{2}$$
$$a_1 \text{ of } \sum n^3 \text{ is } \left(a_1 \text{of} \sum n^2\right) * \frac{2}{2} = \frac{1}{2} * \frac{2}{2} = \frac{1}{2}$$
$$a_1 \text{ of } \sum n^4 \text{ is } \left(a_1 \text{of} \sum n^3\right) * \frac{2}{2} = \frac{1}{2} * \frac{2}{2} = \frac{1}{2}$$

… … … … … … …

Take first a_2 i.e. $\frac{1}{6}$ (of $\sum n^2$)

$$a_2 \text{ of } \sum n^3 \text{ is } \left(a_2 \text{of} \sum n^2\right) * \frac{3}{3-1} = \frac{1}{6} * \frac{3}{2} = \frac{1}{4}$$
$$a_2 \text{ of } \sum n^4 \text{ is } \left(a_2 \text{of} \sum n^3\right) * \frac{4}{4-1} = \frac{1}{4} * \frac{4}{3} = \frac{1}{3}$$
$$a_2 \text{ of } \sum n^5 \text{ is } \left(a_2 \text{of} \sum n^4\right) * \frac{5}{5-1} = \frac{1}{3} * \frac{5}{4} = \frac{5}{12}$$
$$a_2 \text{ of } \sum n^6 \text{ is } \left(a_2 \text{of} \sum n^5\right) * \frac{6}{6-1} = \frac{5}{12} * \frac{6}{5} = \frac{1}{2}$$
$$\sum n^3 = a_0 n^3 + a_1 n^2 + a_2 n + a_3$$
$$\sum n^3 = \frac{1}{4} n^3 + \frac{1}{2} n^2 + \frac{1}{4} n + a_3$$

$$\frac{1}{4} + \frac{1}{2} + \frac{1}{4} + a_3 = 1 \rightarrow a_3 = 0$$

$$\sum n^3 = \frac{1}{4} n^3 + \frac{1}{2} n^2 + \frac{1}{4} n$$

Take first a_3 i.e. 0 (of $\sum n^3$)

$$a_3 \text{ of } \sum n^4 \text{ is } \left(a_3 \text{of } \sum n^3\right) * \frac{4}{4-2} = 0$$

$$a_3 \text{ of } \sum n^5 \text{ is } \left(a_3 \text{of } \sum n^3\right) * \frac{5}{5-2} = 0 \text{ (etc)}$$

Now one can find

$$\sum n^4 = a_0 n^5 + a_1 n^4 + a_2 n^3 + a_3 n^2 + a_4 n$$

$= \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 + 0n^2 + a_4 n$

$a_4 = 1 - \frac{1}{5} - \frac{1}{2} - \frac{1}{3} = \frac{30-6-15-10}{30} = \frac{1}{30}$

$$\sum n^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 + 0n^2 - \frac{1}{30} n$$

Take first a_4 i.e. $-\frac{1}{30}$ (of $\sum n^4$)

$$a_4 \text{ of } \sum n^5 \text{ is } \left(a_4 \text{of } \sum n^4\right) * \frac{5}{5-3} = -\frac{1}{30} * \frac{5}{2} = -\frac{1}{12}$$

$$a_4 \text{ of } \sum n^6 \text{ is } \left(a_4 \text{of } \sum n^5\right) * \frac{6}{6-3} = -\frac{1}{12} * \frac{6}{3} = -\frac{1}{6} \text{(etc).}$$

$$\sum n^5 = a_0 n^6 + a_1 n^5 + a_2 n^4 + a_3 n^3 + a_4 n^2 + a_5 n$$

$= \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 + 0n^3 - \frac{1}{12} n^2 - a_5 n$

$\frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12} + a_5 = 1 \rightarrow a_5 = 0$

$$\sum n^5 = \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 + 0n^3 - \frac{1}{12} n^2 + 0\, n$$

Take first a_5 i.e. 0 (of $\sum n^5$)

a_5 of $\sum n^6$ is ($a_5$of $\sum n^5$) $\left(\frac{6}{6-4}\right) = 0$

$$a_5 \text{of} \sum n^7 = 0 \text{ etc.}$$

Now $\sum n^6 = a_0 n^7 + a_1 n^6 + a_2 n^5 + a_3 n^4 + a_4 n^3 + a_5 n^2 + a_6 n$

$$\sum n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 + 0\, n^4 - \frac{1}{6} n^3 + 0\, n^2 + a_6 n$$

$\frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + a_6 = 1$ which implies $a_6 = \frac{1}{42}$

$$\sum n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 + 0\, n^4 - \frac{1}{6} n^3 + 0\, n^2 + \frac{1}{42} n$$

a_i of $\sum n^r = (a_i \text{ of } \sum n^{r-1}) \left(\frac{r}{r-(i-1)}\right)$where i=0,1,2,…r

5. EXISTENCE AND UNIQUENESS OF POLYNOMIAL

Assume that

$\sum n^k = a_0 n^{k+1} + a_1 n^k + a_2 n^{k-1} + a_3 n^{k-2} + \cdots + a_k n + a_{k+1}$ (1)

$$\sum (n+1)^k = a_0 (n+1)^{k+1} + a_1 (n+1)^k + a_2 (n+1)^{k-1} + a_3 (n+1)^{k-2} + \cdots + a_k(n+1) + a_{k+1}$$

By subtraction

$(n+1)^k = a_0\left[(n+1)^{k+1} - n^{k+1}\right] + a_1\left[(n+1)^k - n^k\right] + a_2\left[(n+1)^{k-1} - n^{k-1}\right] + a_3\left[(n+1)^{k-2} - n^{k-2}\right] + \cdots + a_k[(n+1) - n]$

$(n+1)^k = a_0\left[(k+1)_{c_1} n^k + (k+1)_{c_2} n^{k-1} + \cdots + (k+1)_{c_k} n + 1\right]$

$+a_1\left[k_{c_1} n^{k-1} + k_{c_2} n^{k-2} + \cdots + k_{c_{k-1}} n + 1\right]$

$+a_2\left[(k-1)_{c_1} n^{k-2} + (k-1)_{c_2} n^{k-3} + \cdots + (k-1)_{c_{k-2}} n + 1\right]$

$+a_3\left[(k-2)_{c_1} n^{k-3} + (k-2)_{c_2} n^{k-4} + \cdots + (k-2)_{c_{k-3}} n + 1\right]$

+………………………………………………………………………

+ a_k

But L.H.S.= $k_{c_0} n^k + k_{c_1} n^{k-1} + k_{c_2} n^{k-2} + \cdots + k_{c_{k-1}} n + 1$
Comparing the like powers of n
$k_{c_0} = a_0 (k+1)_{c_1}$
$k_{c_1} = a_0 (k+1)_{c_2} + a_1 k_{c_1}$
$k_{c_2} = a_0 (k+1)_{c_3} + a_1 k_{c_2} + a_2 (k-1)_{c_1}$
………………………………………………..
$k_{c_k} = a_0 (k+1)_{c_{(k+1)}} + a_1 k_{c_k} + a_2 (k-1)_{c_{(k-1)}} + \cdots + a_k 1_{c_1}$…………… (2)
(2) represents a system of non-homogeneous linear equations in (k+1) unknowns $a_0, a_1, a_2 \ldots a_k$

Coefficient Matrix = $\begin{bmatrix} (k+1)_{c_1} & 0 & 0 & . & . & . & 0 \\ (k+1)_{c_2} & k_{c_1} & 0 & . & . & . & 0 \\ (k+1)_{c_3} & k_{c_2} & (k-1)_{c_1} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ (k+1)_{c_{k+1}} & k_{c_k} & (k-1)_{c_{k-1}} & . & . & . & 1_{c_1} \end{bmatrix}$

Variable Matrix= $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_k \end{bmatrix} = X$

Column Matrix = $\begin{bmatrix} K_{c_0} \\ K_{c_1} \\ K_{c_2} \\ . \\ . \\ . \\ K_{c_K} \end{bmatrix} = B$

In matrix notation (2) is written as AX=B.
Augmented Matrix is [AB].
$|A| = (k+1)!$ Since A is lower triangular matrix.
as $|A|$ is non-zero, $\rho(A) = k+1 = A$ is with full rank
$\rho(A+B) = k+1$
Since $\rho(A) = \rho([AB])$= number of unknowns the system (1) has unique solution.
By considering (1) and the last equation of (2) we see that $a_{k+1} = 0$ as shown below.
From last equation of (2) $a_0 + a_1 + a_2 + \cdots a_k = 1$
From (1), for n=1, we get $1 = a_0 + a_1 + a_2 + \cdots a_k + a_{k+1}$
So $a_{k+1} = 0$
Hence we conclude that $\sum n^k$ is always equal to a unique polynomial of degree k+1 in n over natural numbers.

6.CONCLUSION AND FUTURE RESEARCH

The above discussion provides two facts that the sum of positive integral powers of first n-natural numbers coincides with a unique polynomial and such polynomial always exists. Using the patterns described here, the formula to compute the sum of any positive integral powers of first n natural numbers can be easily written .In this research article the applications of Linear Algebra in giving the answers to an ancient interesting problem in research field of Analytical Number Theory have been extensively discussed. In the context of future research these ideas can be proved by using some principles of Integral Calculus and Complex Analysis.

REFERENCES:

[1] Dohyoung Ryang, Tony Thompson (2012), “Sum of positive integral powers “August 2012 Mathematics Teacher 106 (1):71:77.
[2] 2. Janet Beery (2010), “Sum of powers of positive integers –conclusion”, Convergence,July 2010,MAA (Mathematics Teachers America)
[3] 3. Do Tan Si,”The Powers Sums, Bernoulli numbers, Bernoulli Polynomials Rethinked”, Applied mathematics 10.03(2019):100-112, Scientific Research
[4] 4.Reznick, Bruce and Rouse j,”On the sums of two cubes”, International Journal of number theory,7(2011),1863-1882,MR 2854220
[5] 5. Mathematical Masterpieces, by Arthur Knoebel, et al., Springer (2007), ISBN 978-0-387-33060-0. An English translation of the Potestatum appears on pp 32-37 (from the French in the 1923 Grands Écrivains edition). Good discussion is had just before and after the Potestatum itself.
[6] 6."Pascal's Formula for the Sums of Powers of the Integers", by Carl B. Boyer, Scripta Mathematica 9 (1943), pp 237-244. Good historical background on antecedents to Pascal work on sums of powers. Contains an English translation of some of the Potestatum.
[7] 7. Pascal's Arithmetical Triangle, by A. W. F. Edwards, The Johns Hopkins University Press (2002, originally published 1987), ISBN 0-8018-6946-3.